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AN ELEMENTARY TREATISE ON

QUATERNIONS

BY

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τετρακτύν,
παγὰν ἀενάου φύσεως ῥιζώματ' ἔχουσαν.

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PREFACE.

The present work was commenced in 1859, while I was a Professor of Mathematics, and far more ready at Quaternion analysis than I can now pretend to be. Had it been then completed I should have had means of testing its teaching capabilities, and of improving it, before publication, where found deficient in that respect.

The duties of another Chair, and Sir W. Hamilton's wish that my volume should not appear till after the publication of his *Elements*, interrupted my already extensive preparations. I had worked out nearly all the examples of Analytical Geometry in Todhunter's Collection, and I had made various physical applications of the Calculus, especially to Crystallography, to Geometrical Optics, and to the Induction of Currents, in addition to those on Kinematics, Electrodynamics, &c., which are reprinted in the present work from the *Quarterly Mathematical Journal* and the *Proceedings of the Royal Society of Edinburgh.*

Sir W. Hamilton, a few days before his death, urged me to prepare my work as soon as possible, his being almost ready for publication. He then expressed, more strongly perhaps than he had ever done before, his profound conviction of the importance of Quaternions to the progress of physical science; and his desire that a really elementary treatise on the subject should soon be published.

I regret that I have so imperfectly fulfilled this last request of my revered friend. When it was made I was already engaged, along with Sir W. Thomson, in the laborious work of preparing a large Treatise on Natural Philosophy. The present volume has thus been written under very disadvantageous circumstances, especially as I have not found time to work up the mass of materials which I had originally collected for it, but which I had not put into a fit state for publication. I hope, however, that I have to some extent succeeded in producing a thoroughly elementary work, intelligible to any ordinary student; and that the numerous examples I have given, though not specially chosen so as to display the full merits of Quaternions, will yet sufficiently show their admirable simplicity and naturalness to induce the reader to attack the *Lectures* and the *Elements;* where he will find, in profusion, stores of valuable results, and of elegant yet powerful analytical investigations, such as are contained in the writings of but a very few of the greatest mathematicians. For a succinct account of the steps by which Hamilton was led to the invention of Quaternions, and for other interesting information regarding that remarkable genius, I may refer to a slight sketch of his life and works in the *North British Review* for September 1866.

It will be found that I have not servilely followed even so great a master, although dealing with a subject which is entirely his own. I cannot, of course, tell in every case what I have gathered from his published papers, or from his voluminous correspondence, and what I may have made out for myself. Some theorems and processes which I have given, though wholly my own, in the sense of having been made out for myself before the publication of the *Elements*, I have since found there. Others also may be, for I have not yet read that tremendous volume completely, since much of it bears on developments unconnected with Physics. But I have endeavoured throughout to point out to the reader all the more important parts of the work which I know to be wholly due to Hamilton. A great part, indeed, may be said to be obvious to any one who has mastered the preliminaries; still I think that, in the two last Chapters especially, a good deal of original matter will be found.

The volume is essentially a *working* one, and, especially in the later Chapters, is rather a collection of examples than a detailed treatise on a mathematical method. I have constantly aimed at avoiding too great extension; and in pursuance of this object have omitted many valuable elementary portions of the subject. One of these, the treatment of Quaternion logarithms and exponentials, I greatly regret not having given. But if I had printed all that seemed to me of use or interest to the student, I might easily have rivalled the bulk of one of Hamilton's volumes. The beginner is recommended merely to *read* the first five Chapters, then to *work* at Chapters VI, VII, VIII (to which numerous easy Examples are appended). After this he may work at the first five, with their (more difficult) Examples; and the remainder of the book should then present no difficulty.

Keeping always in view, as the great end of every mathematical method, the physical applications, I have endeavoured to treat the subject as much as possible from a geometrical instead of an analytical point of view. Of course, if we premise the properties of i, j, k merely, it is possible to construct from them the whole system*; just as we deal with the imaginary of

---

* This has been done by Hamilton himself, as one among many methods he has employed; and it is also the foundation of a memoir by M. Allégret entitled *Essai sur le Calcul des Quaternions* (Paris, 1862).

Algebra, or, to take a closer analogy, just as Hamilton himself dealt with Couples, Triads, and Sets. This may be interesting to the pure analyst, but it is repulsive to the physical student, who should be led to look upon i, j, k from the very first as geometric realities, not as algebraic imaginaries.

The most striking peculiarity of the Calculus is that *multiplication is not generally commutative*, i. e. that qr is in general different from rq, r and q being quaternions. Still it is to be remarked that something similar is true, in the ordinary cöordinate methods, of operators and functions: and therefore the student is not wholly unprepared to meet it. No one is puzzled by the fact that $\log.\cos.x$ is not equal to $\cos.\log.x$, or that $\sqrt{\frac{dy}{dx}}$ is not equal to $\frac{d}{dx}\sqrt{y}$. Sometimes, indeed, this rule is most absurdly violated, for it is usual to take $\cos^2 x$ as equal to $(\cos x)^2$, while $\cos^{-1} x$ is not equal to $(\cos x)^{-1}$. No such incongruities appear in Quaternions; but what is true of operators and functions in other methods, that they are not generally commutative, is in Quaternions true in the multiplication of (vector) cöordinates.

It will be observed by those who are acquainted with the Calculus that I have, in many cases, not given the shortest or simplest proof of an important proposition. This has been done with the view of including, in moderate compass, as great a variety of methods as possible. With the same object I have endeavoured to supply, by means of the Examples appended to each Chapter, hints (which will not be lost to the intelligent student) of farther developments of the Calculus. Many of these are due to Hamilton, who, in spite of his great originality, was one of the most excellent examiners any University can boast of.

It must always be remembered that Cartesian methods are mere particular cases of Quaternions, where most of the distinctive features have disappeared; and that when, in the treatment of any particular question, scalars have to be adopted, the Quaternion solution becomes identical with the Cartesian one. Nothing therefore is ever lost, though much is generally gained, by employing Quaternions in preference to ordinary methods. In fact, even when Quaternions degrade to scalars, they give the solution of the most general statement of the problem they are applied to, quite independent of any limitations as to choice of particular cöordinate axes.

There is one very desirable object which such a work as this may possibly fulfil. The University of Cambridge, while seeking to supply a real want (the deficiency of subjects of examination for mathematical honours, and the consequent frequent introduction of the wildest extravagance in the shape of data for "Problems"), is in danger of making too much of such elegant trifles as Trilinear Cöordinates, while gigantic systems like Invariants (which, by the way, are as easily introduced into Quaternions as into Cartesian methods) are quite beyond the amount of mathematics which even the best students can master in three years' reading. One grand step to the supply of this want is, of course, the introduction into the scheme of examination of such branches of mathematical physics as the Theories of Heat and Electricity. But it appears to me that the study of a mathematical method like Quaternions, which, while of immense power and comprehensiveness, is of extraordinary simplicity, and yet requires constant thought in its applications, would also be of great benefit. With it there can be no "shut your eyes, and write down your equations," for mere mechanical dexterity of analysis is certain to lead at once to error on account of the novelty of the processes employed.

The Table of Contents has been drawn up so as to give the student a short and simple summary of the chief fundamental formulae of the Calculus itself, and is therefore confined to an analysis of the first five chapters.

In conclusion, I have only to say that I shall be much obliged to any one, student or teacher, who will point out portions of the work where a difficulty has been found; along with any inaccuracies which may be detected. As I have had no assistance in the revision of the proof-sheets, and have composed the work at irregular intervals, and while otherwise laboriously occupied, I fear it may contain many slips and even errors. Should it reach another edition there is no doubt that it will be improved in many important particulars.

P. G. TAIT.

COLLEGE, EDINBURGH,
July 1867.

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$$\rho = \alpha + x\beta$$

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$$\rho = y\alpha + x\beta$$

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$$p + q + r = 0.$$

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$$\alpha\beta = K\beta\alpha,$$
$$S.qrs = S.srq,$$
$$S.\alpha\beta\gamma = S.\beta\gamma\alpha = S.\gamma\alpha\beta = -S.\alpha\gamma\beta = \text{ \&c. } \quad \S\S\ 86\text{–}89.$$

Proof of the formulae

$$V.\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\gamma\alpha,$$
$$V.\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta,$$
$$V.\alpha\beta\gamma = V.\gamma\beta\alpha,$$
$$\delta S.\alpha\beta\gamma = \alpha S.\beta\gamma\delta + \beta S.\gamma\alpha\delta + \gamma S.\alpha\beta\delta,$$
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$$TV\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}\sin\theta, \qquad TV\alpha\beta = T\alpha T\beta\sin\theta.$$

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shows that α is perpendicular to β, while

$$V\alpha\beta = 0,$$

shows that α and β are parallel.

$$S.\alpha\beta\gamma$$

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$$S.\alpha\beta\gamma = 0$$

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$$(S\alpha\rho)^2+(S\beta\rho)^2+(S\gamma\rho)^2=\left(\frac{T(\iota\rho+\rho\kappa)}{\kappa^2-\iota^2}\right)^2,$$

$$\text{where } 2\left\{\begin{matrix}\iota\\ \kappa\end{matrix}\right\}=\frac{T\alpha\pm T\gamma}{T\alpha T\gamma}\left(\sqrt{\frac{\beta^2-\alpha^2}{\gamma^2-\alpha^2}}U\alpha\pm\sqrt{\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2}}U\gamma\right). \quad \text{§§ } 121, 122.$$

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where $$m_1 = \frac{S(\lambda\phi'\mu\phi'\nu + \phi'\lambda\mu\phi'\nu + \phi'\lambda\phi'\mu\nu)}{S.\lambda\mu\nu},$$

and $$m_2 = \frac{S(\lambda\mu\phi'\nu + \phi'\lambda\mu\nu + \lambda\phi'\mu\nu)}{S.\lambda\mu\nu},$$

then $$m_g(\phi + g)^{-1} V\lambda\mu = (m\phi^{-1} + g\chi + g^2) V\lambda\mu.$$

Also that $$\chi = m_2 - \phi,$$

whence the final form of solution

$$m\phi^{-1} = m_1 - m_2\phi + \phi^2, \text{ §§ 147, 148.}$$

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The fundamental cubic

$$\phi^3 - m_2\phi^2 + m_1\phi - m = (\phi - g_1)(\phi - g_2)(\phi - g_3) = 0.$$

When ϕ is its own conjugate, the roots of the cubic are real; and the equation $$V\rho\phi\rho = 0,$$

or $$(\phi - g)\rho = 0,$$

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$$\phi\rho = p\rho + q V.(i + ek)\rho(i - ck)$$

where $$(\phi - g_1)i = 0,$$
$$(\phi - g_3)k = 0,$$

$$e^2 = \frac{g_2 - g_3}{g_1 - g_2},$$
$$p = \tfrac{1}{2}(g_1 + g_2),$$
$$q = -\tfrac{1}{2}(g_1 - g_2).$$

Another transformation is

$$\phi\rho = a\alpha V\alpha\rho + b\beta S\beta\rho, \quad \S\S\ 167\text{–}169.$$

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$$S\rho(\phi + g)^{-1}\rho = 0, \quad \text{and} \quad S\rho(\phi + h)^{-1}\rho = 0$$

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ERRATA.

Page 171, first line of § 249, *for* OD *read* $\overline{OD}$.

,, 213, last line but one, *for* $S\alpha\beta\rho$ *read* $S.\alpha\beta\rho$.

,, 225, line 10, *for* equations *read* equation.

QUATERNIONS.

CHAPTER I.

VECTORS, AND THEIR COMPOSITION.

1. FOR more than a century and a half the geometrical representation of the negative and imaginary algebraic quantities, -1 and $\sqrt{-1}$, or, as some prefer to write them, $-$ and $-^{\frac{1}{2}}$, has been a favourite subject of speculation with mathematicians. The essence of almost all of the proposed processes consists in employing such quantities to indicate the *direction*, not the *length*, of lines.

2. Thus it was soon seen that if positive quantities were measured off in one direction along a fixed line, a useful and lawful convention enabled us to express negative quantities by simply laying them off on the same line in the opposite direction. This convention is an essential part of the Cartesian method, and is constantly employed in Analytical Geometry and Applied Mathematics.

3. Wallis, in the end of the seventeenth century, proposed to represent the impossible roots of a quadratic equation by going *out of* the line on which, if real, they would have been laid off.

His construction is equivalent to the consideration of $\sqrt{-1}$ as a directed unit-line perpendicular to that on which real quantities are measured.

4. In the usual notation of Analytical Geometry of two dimensions, when rectangular axes are employed, this amounts to reckoning each unit of length along Oy as $+\sqrt{-1}$, and on Oy' as $-\sqrt{-1}$; while on Ox each unit is $+1$, and on Ox' it is -1. If we look at these four lines in circular order, i. e. in the order of positive rotation (opposite to that of the hands of a watch), they give

$$1, \quad \sqrt{-1}, \quad -1, \quad -\sqrt{-1}.$$

In this series each expression is derived from that which precedes it by multiplication by the factor $\sqrt{-1}$. Hence we may consider $\sqrt{-1}$ as an operator, analogous to a handle perpendicular to the plane of xy, whose effect on any line is to make it rotate (positively) about the origin through an angle of 90°.

5. In such a system, a point is defined by a single imaginary expression. Thus $a+b\sqrt{-1}$ may be considered as a single quantity, denoting the point whose cöordinates are a and b. Or, it may be used as an expression for the line joining that point with the origin. In the latter sense, the expression $a+b\sqrt{-1}$ implicitly contains the *direction*, as well as the *length*, of this line; since, as we see at once, the direction is inclined at an angle $\tan^{-1}\frac{b}{a}$ to the axis of x, and the length is $\sqrt{a^2+b^2}$.

6. Operating on this symbol by the factor $\sqrt{-1}$, it becomes $-b+a\sqrt{-1}$; and now, of course, denotes the point whose x and y cöordinates are $-b$ and a; or the line joining this point with the origin. The length is still $\sqrt{a^2+b^2}$, but the angle the line makes with the axis of x is $\tan^{-1}\left(-\frac{a}{b}\right)$; which is evidently 90° greater than before the operation.

7. De Moivre's Theorem tends to lead us still farther in the same direction. In fact, it is easy to see that if we use, instead of $\sqrt{-1}$, the more general factor $\cos a + \sqrt{-1} \sin a$, its effect on any line is to turn it through the (positive) angle a in the plane of x, y. [Of course the former factor, $\sqrt{-1}$, is merely the particular case of this, when $a = \frac{\pi}{2}$.]

Thus $(\cos a + \sqrt{-1} \sin a)(a + b\sqrt{-1})$

$$= a \cos a - b \sin a + \sqrt{-1}\,(a \sin a + b \cos a),$$

by direct multiplication. The reader will at once see that the new form indicates that a rotation through an angle a has taken place, if he compares it with the common formulæ for turning the cöordinate axes through a given angle. Or, in a less simple manner, thus—

$$\begin{aligned} \text{Length} &= \sqrt{(a \cos a - b \sin a)^2 + (a \sin a + b \cos a)^2} \\ &= \sqrt{a^2 + b^2} \quad \text{as before.} \end{aligned}$$

Inclination to axis of x

$$= \tan^{-1} \frac{a \sin a + b \cos a}{a \cos a - b \sin a} = \tan^{-1} \frac{\tan a + \frac{b}{a}}{1 - \frac{b}{a} \tan a}$$

$$= a + \tan^{-1} \frac{b}{a}.$$

8. We see now, as it were, *why* it happens that

$$(\cos a + \sqrt{-1} \sin a)^m = \cos ma + \sqrt{-1} \sin ma.$$

In fact, the first operator produces m successive rotations in the same direction, each through the angle a; the second, a single rotation through the angle ma.

9. It may be interesting, at this stage, to anticipate so far as to state that a Quaternion can, in general, be put under the form

$$N(\cos \theta + \varpi \sin \theta),$$

where N is a numerical quantity, θ a real angle, and

$$\varpi^2 = -1.$$

This expression for a quaternion bears a very close analogy to the forms employed in De Moivre's Theorem; but there is the essential difference (to which Hamilton's chief invention referred) that ϖ is not the algebraic $\sqrt{-1}$, but may be *any directed unit-line* whatever in space.

10. In the present century Argand, Warren, and others, extended the results of Wallis and De Moivre. They attempted to express as a line the product of two lines each represented by a symbol such as $a+b\sqrt{-1}$. To a certain extent they succeeded, but simplicity was not gained by their methods, as the terrible array of radicals in Warren's Treatise sufficiently proves.

11. A very curious speculation, due to Servois, and published in 1813 in Gergonne's *Annales*, is the only one, so far as has been discovered, in which the slightest trace of an anticipation of Quaternions is contained. Endeavouring to extend to *space* the form $a+b\sqrt{-1}$ for the plane, he is guided by analogy to write for a directed unit-line in space the form

$$p\cos\alpha+q\cos\beta+r\cos\gamma,$$

where α, β, γ are its inclinations to the three axes. He perceives easily that p, q, r must be *non-reals:* but, he asks, "seraient-elles *imaginaires* réductibles à la forme générale $A+B\sqrt{-1}$?" This he could not answer. In fact they are the i, j, k of the Quaternion Calculus. (See Chap. II.)

12. Beyond this, few attempts were made, or at least recorded, in earlier times, to extend the principle to space of three dimensions; and, though many such have been made within the last forty years, none, with the single exception of Hamilton's, have resulted in simple, practical methods; all, however ingenious, seeming to lead at once to processes and results of fearful complexity.

For a lucid, complete, and most impartial statement of the claims of his predecessors in this field we refer to the Preface to Hamilton's *Lectures on Quaternions.*

13. It was reserved for Hamilton to discover the use of $\sqrt{-1}$ as a *geometric reality,* tied down to no particular direction in space, and this use was the foundation of the singularly elegant, yet enormously powerful, Calculus of Quaternions.

While all other schemes for using $\sqrt{-1}$ to indicate direction make one direction in space expressible by real numbers, the remainder being imaginaries of some kind, leading in general to equations which are heterogeneous; Hamilton makes all directions in space equally imaginary, or rather equally real, thereby ensuring to his Calculus the power of dealing with space indifferently in all directions.

In fact, as we shall see, the Quaternion method is independent of axes or any supposed directions in space, and takes its reference lines solely from the problem it is applied to.

14. But, for the purpose of elementary exposition, it is best to begin by assimilating it as closely as we can to the ordinary Cartesian methods of Geometry of Three Dimensions, which are in fact a mere particular case of Quaternions in which most of the distinctive features are lost. We shall find in a little that it is capable of soaring above these entirely, after having employed them in its establishment; and, indeed, as the inventor's works amply prove, it can be established, *ab initio,* in various ways, without even an allusion to Cartesian Geometry. As this work is written for students acquainted with at least the elements of the Cartesian method, we keep to the first-mentioned course of exposition; especially as we thereby avoid some reasoning which, though rigorous and beautiful, might be apt, from its subtlety, to prove repulsive to the beginner.

We commence, therefore, with some very elementary geometrical ideas.

15. Suppose we have two points A and B in *space*, and suppose A given, on how many numbers does B's relative position depend?

If we refer to Cartesian cöordinates (rectangular or not) we find that the data required are the excesses of B's three cöordinates over those of A. Hence *three* numbers are required.

Or we may take polar cöordinates. To define the moon's position with respect to the earth we must have its Geocentric Latitude and Longitude, *or* its Right Ascension and Declination, and, in addition, its distance or radius-vector. *Three* again.

16. Here it is to be carefully noticed that nothing has been said of the *actual* cöordinates of either A or B, or of the earth and moon, in space; it is only the *relative* cöordinates that are contemplated.

Hence any expression, as $\overline{AB}$, denoting a line considered with reference to direction as well as length, contains implicitly *three* numbers, and all lines parallel and equal to AB depend in the same way upon the same three. Hence, *all lines which are equal and parallel may be represented by a common symbol, and that symbol contains three distinct numbers.* In this sense a line is called a VECTOR, since by it we pass from the one extremity, A, to the other, B; and it may thus be considered as an instrument which *carries* A to B: so that a vector may be employed to indicate a definite *translation* in space.

17. We may here remark, once for all, that in establishing a new Calculus, we are at liberty to give any definitions whatever of our symbols, provided that no two of these interfere with, or contradict, each other, and in doing so in Quaternions *simplicity* and (so to speak) *naturalness* were the inventor's aim.

18. Let $\overline{AB}$ be represented by α, we know that α depends on *three* separate numbers. Now if CD be equal in length to AB

and if these lines be parallel, we have evidently $\overline{CD} = \overline{AB} = \alpha$, where it will be seen that the sign of *equality* between vectors contains implicitly *equality in length* and *parallelism in direction*. So far we have *extended* the meaning of an algebraic symbol. And it is to be noticed that an equation between vectors, as

$$\alpha = \beta,$$

contains *three* distinct equations between mere numbers.

19. We must now define + (and the meaning of − will follow) in the new Calculus. Let A, B, C be any three points and (with the above meaning of $=$) let

$$\overline{AB} = \alpha, \quad \overline{BC} = \beta, \quad \overline{AC} = \gamma.$$

If we define + (in accordance with the idea (§ 16) that a vector represents a *translation*) by the equation

$$\alpha + \beta = \gamma,$$

or

$$\overline{AB} + \overline{BC} = \overline{AC},$$

we contradict nothing that precedes, but we at once introduce the idea that *vectors are to be compounded, in direction and magnitude, like simultaneous velocities.* A reason for this may be seen in another way if we remember that by *adding* the differences of the Cartesian coördinates of A and B, to those of the coördinates of B and C, we get those of the coördinates of A and C.

20. But we also see that if C and A coincide (and C may be *any* point)

$$\overline{AC} = 0,$$

for no vector is then required to carry A to C. Hence the above relation may be written, in this case,

$$\overline{AB} + \overline{BA} = 0,$$

or, introducing, and by the same act defining, the symbol −,

$$\overline{BA} = -\overline{AB}.$$

Hence, *the symbol* $-$, *applied to a vector, simply shows that its direction is to be reversed.*

And this is consistent with all that precedes; for instance,

$$\overline{AB} + \overline{BC} = \overline{AC},$$

and

$$\overline{AB} = \overline{AC} - \overline{BC},$$

or

$$= \overline{AC} + \overline{CB},$$

are evidently but different expressions of the same truth.

21. In any triangle, ABC, we have, of course,

$$\overline{AB} + \overline{BC} + \overline{CA} = 0;$$

and, in any closed polygon, whether plane or gauche,

$$\overline{AB} + \overline{BC} + \ldots\ldots + \overline{YZ} + \overline{ZA} = 0.$$

In the case of the polygon we have also

$$\overline{AB} + \overline{BC} + \ldots\ldots + \overline{YZ} = \overline{AZ}.$$

These are the well-known propositions regarding composition of velocities, which, by the second law of motion, give us the geometrical laws of composition of forces.

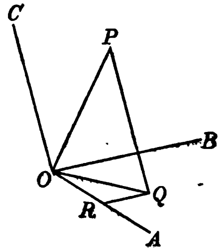
22. If we compound any number of *parallel* vectors, the result is obviously a numerical multiple of any one of them.

Thus, if A, B, C are in one straight line,

$$\overline{BC} = x\overline{AB};$$

where x is a number, positive when B lies between A and C, otherwise negative: but such that its numerical value, independent of sign, is the ratio of the length of BC to that of AB. This is at once evident if AB and BC be commensurable; and is easily extended to incommensurables by the usual *reductio ad absurdum.*

23. An important, but almost obvious, proposition is that *any vector may be resolved into three components parallel respectively* *to any three given vectors, no two of which are parallel, and which are not parallel to one plane.*



Let OA, OB, OC be the three fixed vectors, OP any other vector. From P draw PQ parallel to CO, meeting the plane BOA in Q. [There must be a real point Q, else PQ, and therefore CO, would be parallel to BOA, a case specially excepted.] From Q draw QR parallel to BO, meeting OA in R. Then we have

$$\overline{OP} = \overline{OR} + \overline{RQ} + \overline{QP} \quad (\S\ 21),$$

and these components are respectively parallel to the three given vectors. By § 22 we may express $\overline{OR}$ as a numerical multiple of $\overline{OA}$, $\overline{RQ}$ of $\overline{OB}$, and $\overline{QP}$ of $\overline{OC}$. Hence we have, generally, for any vector in terms of three fixed non-coplanar vectors, α, β, γ,

$$\overline{OR} = \rho = x\alpha + y\beta + z\gamma,$$

which exhibits, in one form, the *three* numbers on which a vector depends (§ 16). Here x, y, z are perfectly definite.

24. Similarly any vector, as $\overline{OQ}$, in the same plane with OA and OB, can be resolved into components $\overline{OR}$, $\overline{RQ}$, parallel respectively to $\overline{OA}$ and $\overline{OB}$; so long, at least, as these two vectors are not parallel to each other.

25. There is particular advantage, in certain cases, in employing a series of three *mutually perpendicular unit-vectors* as lines of reference. This system Hamilton denotes by i, j, k.

Any other vector is then expressible as

$$\rho = xi + yj + zk.$$

Since i, j, k are unit-vectors, x, y, z are here the lengths of conterminous edges of a rectangular parallelepiped of which ρ is the vector-diagonal; so that the length of ρ is, in this case,

$$\sqrt{x^2 + y^2 + z^2}.$$

Let
$$\varpi = \xi i + \eta j + \zeta k$$
be any other vector, then the vector equation
$$\rho = \varpi$$
obviously involves the following three equations among numbers,
$$x = \xi, \quad y = \eta, \quad z = \zeta.$$
Suppose i to be drawn eastwards, j northwards, and k upwards, this is equivalent merely to saying that *if two points coincide, they are equally to the east* (*or west*) *of any third point, equally to the north* (*or south*) *of it, and equally elevated above* (*or depressed below*) *its level.*

26. It is to be carefully noticed that it is only when α, β, γ are not coplanar that a vector equation such as
$$\rho = \varpi,$$
or
$$x\alpha + y\beta + z\gamma = \xi\alpha + \eta\beta + \zeta\gamma,$$
necessitates the three numerical equations
$$x = \xi, \quad y = \eta, \quad z = \zeta.$$
For, if α, β, γ be coplanar (§ 24), a condition of the following form must hold
$$\gamma = a\alpha + b\beta.$$
Hence
$$\rho = (x + za)\alpha + (y + zb)\beta,$$
$$\varpi = (\xi + \zeta a)\alpha + (\eta + \zeta b)\beta,$$
and the equation
$$\rho = \varpi$$
now requires only the *two* numerical conditions
$$x + za = \xi + \zeta a, \qquad y + zb = \eta + \zeta b.$$

27. *The Commutative and Associative Laws hold in the combination of vectors by the signs* $+$ *and* $-$. It is obvious that, if we prove this for the sign $+$, it will be equally proved for $-$, because $-$ before a vector (§ 20) merely indicates that it is to be reversed before being considered positive.

Let A, B, C, D be, in order, the corners of a parallelogram; we have, obviously,

$$\overline{AB} = \overline{DC}, \quad \overline{AD} = \overline{BC}.$$

And $\overline{AB}+\overline{BC} = \overline{AC} = \overline{AD}+\overline{DC} = \overline{BC}+\overline{AB}.$

Hence the commutative law is true for the addition of any two vectors, and is therefore generally true.

Again, whatever four points are represented by A, B, C, D, we have

$$\overline{AD} = \overline{AB}+\overline{BD} = \overline{AC}+\overline{CD}, \text{ or}$$

$$\overline{AB}+\overline{BC}+\overline{CD} = \overline{AB}+(\overline{BC}+\overline{CD}) = (\overline{AB}+\overline{BC})+\overline{CD}.$$

And thus the truth of the associative law is evident.

28. The equation

$$\rho = x\beta,$$

where ρ is the vector connecting a variable point with the origin, β a definite vector, and x an indefinite number, represents the straight line drawn from the origin parallel to β (§ 22).

The straight line drawn from A, where $\overline{OA} = \alpha$, and parallel to β, has the equation

$$\rho = \alpha+x\beta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In words, we may pass directly from O to P by the vector $\overline{OP}$ or ρ; or we may pass first to A, by means of $\overline{OA}$ or α, and then to P along a vector parallel to β (§ 16).

Equation (1) is one of the many useful forms into which Quaternions enable us to throw the general equation of a straight line in space. As we have seen (§ 25) it is equivalent to *three* numerical equations; but, as these involve the indefinite quantity x, they are virtually equivalent to but *two*, as in ordinary Geometry of Three Dimensions.

29. A good illustration of this remark is furnished by the fact that the equation

$$\rho = y\alpha+x\beta,$$

which contains two indefinite quantities, is virtually equivalent to only one numerical equation. And it is easy to see that it represents the plane in which the lines α and β lie; or the surface which is formed by drawing, through every point of OA, a line parallel to OB. In fact, the equation, as written, is simply § 24 in symbols.

And it is evident that the equation

$$\rho = \gamma + y\alpha + x\beta$$

is the equation of a plane passing through the extremity of γ, and parallel to α and β.

It will now be obvious to the reader that the equation

$$\rho = p_1\alpha_1 + p_2\alpha_2 + \ldots\ldots = \Sigma p\alpha,$$

where α_1, α_2, &c. are given vectors, and p_1, p_2, &c. numerical quantities, *represents a straight line* if p_1, p_2, &c. be linear functions of *one* indeterminate number; and a *plane*, if they be linear expressions containing *two* indeterminate numbers. Later (§ 31 (*l*)), this theorem will be much extended.

30. The equation of the line joining any two points A and B, where $\overline{OA} = \alpha$ and $\overline{OB} = \beta$, is obviously

$$\rho = \alpha + x(\beta - \alpha),$$

or

$$\rho = \beta + y(\alpha - \beta).$$

These equations are of course identical, as may be seen by putting $1 - y$ for x.

The first may be written

$$\rho + (x-1)\alpha - x\beta = 0;$$

or

$$p\rho + q\alpha + r\beta = 0,$$

subject to the condition $p + q + r = 0$ identically. That is—A homogeneous linear function of three vectors, equated to zero, expresses that the extremities of these vectors are in one straight line, *if the sum of the coefficients be identically zero.*

Similarly, the equation of the plane containing the extremities A, B, C of the three non-coplanar vectors α, β, γ is

$$\rho = \alpha + x(\beta - \alpha) + y(\gamma - \beta),$$

where x and y are each indeterminate.

This may be written

$$p\rho+q\alpha+r\beta+s\gamma = 0,$$

with the identical relation

$$p+q+r+s = 0,$$

which is the condition that four points may lie in one plane.

31. We have already the means of proving, in a very simple manner, numerous classes of propositions in plane and solid geometry. A very few examples, however, must suffice at this stage; since we have hardly, as yet, crossed the threshold of the subject, and are dealing with mere linear equations connecting two or more vectors, and even with them *we are restricted as yet to operations of mere addition.*

(*a.*) *The diagonals of a parallelogram bisect each other.*

Let $ABCD$ be the parallelogram, O the point of intersection of its diagonals. Then

$$\overline{AO}+\overline{OB} = \overline{AB} = \overline{DC} = \overline{DO}+\overline{OC},$$

which gives

$$\overline{AO}-\overline{OC} = \overline{DO}-\overline{OB}.$$

The two vectors here equated are parallel to the diagonals respectively. Such an equation is, of course, absurd unless

(1) The diagonals are parallel, in which case the figure is not a parallelogram;

(2) $\overline{AO} = \overline{OC}$, and $\overline{DO} = \overline{OB}$, the proposition.

(*b.*) *To show that a triangle can be constructed, whose sides are parallel, and equal, to the bisectors of the sides of any triangle.*

Let ABC be the triangle, Aa, Bb, Cc the bisectors of the sides. Then

$$\overline{Aa} = \overline{AB} + \overline{Ba} = \overline{AB} + \tfrac{1}{2}\overline{BC},$$

$$\overline{Bb} \quad - \; - \; - \quad = \overline{BC} + \tfrac{1}{2}\overline{CA},$$

$$\overline{Cc} \quad - \; - \; - \quad = \overline{CA} + \tfrac{1}{2}\overline{AB}.$$

Hence $\overline{Aa} + \overline{Bb} + \overline{Cc} = \tfrac{3}{2}(\overline{AB} + \overline{BC} + \overline{CA}) = 0$;
which (§ 21) proves the proposition.

Also
$$\begin{aligned}\overline{Aa} &= \overline{AB} + \tfrac{1}{2}\overline{BC} \\ &= \overline{AB} - \tfrac{1}{2}(\overline{CA} + \overline{AB}) \\ &= \tfrac{1}{2}(\overline{AB} - \overline{CA}) = \tfrac{1}{2}(\overline{AB} + \overline{AC}),\end{aligned}$$

results which are sometimes useful. They may be easily verified by producing Aa to twice its length and joining the extremity with B.

(b'.) *The bisectors of the sides of a triangle meet in a point, which trisects each of them.*

Taking A as origin, and putting α, β, γ for vectors parallel, and equal, to the sides taken in order; the equation of Bb is (§ 28 (1))

$$\rho = \gamma + x\left(\gamma + \frac{\beta}{2}\right) = (1+x)\gamma + \frac{x}{2}\beta.$$

That of Cc is, in the same way,

$$\rho = -(1+y)\beta - \frac{y}{2}\gamma.$$

At the point O, where Bb and Cc intersect,

$$\rho = (1+x)\gamma + \frac{x}{2}\beta = -(1+y)\beta - \frac{y}{2}\gamma.$$

Since γ and β are not parallel, this equation gives

$$1+x = -\frac{y}{2}, \text{ and } \frac{x}{2} = -(1+y).$$

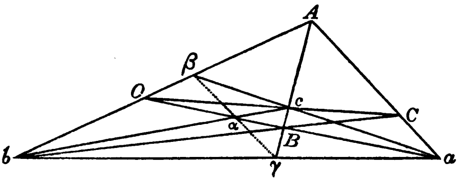
From these $x = y = -\tfrac{2}{3}$.

Hence $\overline{AO} = \tfrac{1}{3}(\gamma - \beta) = \tfrac{2}{3}\overline{Aa}$. (See Ex. ($b$).)

This equation shows, being a vector one, that Aa passes through O, and that $AO : Oa :: 2 : 1$.

(*c.*) *If* $\overline{OA} = \alpha$, $\overline{OB} = \beta$, $\overline{OC} = a\alpha + b\beta$, *be three given co-planar vectors, and the lines indicated in the figure be drawn, the points* α, β, γ *lie in a straight line.*

We see, at once, by the process indicated in § 30, that



$$\overline{Oc} = \frac{a\alpha + b\beta}{a+b},$$

$$\overline{Ob} = \frac{a\alpha}{1-b},$$

$$\overline{Oa} = \frac{b\beta}{1-a}.$$

Hence we easily find

$$\overline{O\alpha} = -\frac{b\beta}{1-a-2b}, \quad \overline{O\beta} = -\frac{a\alpha}{1-2a-b}, \quad \overline{O\gamma} = \frac{-a\alpha + b\beta}{b-a}.$$

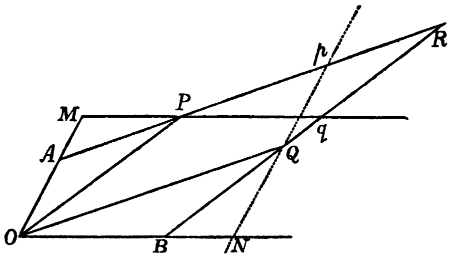
These give

$$-(1-a-2b)\overline{O\alpha} + (1-2a-b)\overline{O\beta} - (b-a)\overline{O\gamma} = 0.$$

But $\quad -(1-a-2b)+(1-2a-b)-(b-a) = 0$ identically.

This, by § 30, proves the proposition.

(*d.*) Let $\overline{OA} = \alpha$, $\overline{OB} = \beta$, be any two vectors. *If MP be parallel to OB; and OQ, BQ, be drawn parallel to AP, OP respectively; the locus of Q is a straight line parallel to OA.*



Let $\overline{OM} = e\alpha$.

Then

$$\overline{AP} = \overline{e-1}\,\alpha + x\beta.$$

Hence the equation of OQ is

$$\rho = y(\overline{e-1}\,\alpha + x\beta);$$

and that of BQ is

$$\rho = \beta + z(e\alpha + x\beta).$$

At Q we have, therefore,

$$\left.\begin{aligned} xy &= 1+zx, \\ y(e-1) &= ze. \end{aligned}\right\}$$

These give $xy = e$, and the equation of the locus of Q is

$$\rho = e\beta + y'\alpha,$$

i. e. a straight line parallel to OA, drawn through N in OB produced, so that

$$ON : OB :: OM : OA.$$

COR. If BQ meet MP in q, $\overline{Pq} = \beta$; and if AP meet NQ in p, $\overline{Qp} = \alpha$.

Also, for the point R we have $\overline{pR} = \overline{AP}$, $\overline{QR} = \overline{Bq}$.

Hence *if from any two points, A and B, lines be drawn intercepting a given length Pq on a given line Mq; and if, from R their point of intersection, Rp be laid off $= PA$, and $RQ = qB$; Q and p lie on a fixed straight line, and the length of Qp is constant.*

(*e.*) *To find the centre of inertia of any system.*

If $\overline{OA} = \alpha$, $\overline{OB} = \alpha_1$, be the vector sides of any triangle, the vector from the vertex dividing the base AB in C so that

$$BC : CA :: m : m_1 \text{ is}$$

$$\frac{m\alpha + m_1\alpha_1}{m+m_1}.$$

For $\overline{AB}$ is $\alpha_1 - \alpha$, and therefore $\overline{AC}$ is

$$\frac{m_1}{m+m_1}(\alpha_1 - \alpha).$$

Hence
$$\begin{aligned} \overline{OC} &= \overline{OA} + \overline{AC} \\ &= \alpha + \frac{m_1}{m+m_1}(\alpha_1 - \alpha) \\ &= \frac{m\alpha + m_1\alpha_1}{m+m_1}. \end{aligned}$$

This expression shows how to find the centre of inertia of two masses; m at the extremity of α, m_1 at that of α_1. Introduce m_2 at the extremity of α_2, then the vector of the centre of inertia of the three is, by a second application of the formula,

$$\frac{(m+m_1)\left(\frac{m\alpha+m_1\alpha_1}{m+m_1}\right)+m_2\alpha_2}{(m+m_1)+m_2}=\frac{m\alpha+m_1\alpha_1+m_2\alpha_2}{m+m_1+m_2}.$$

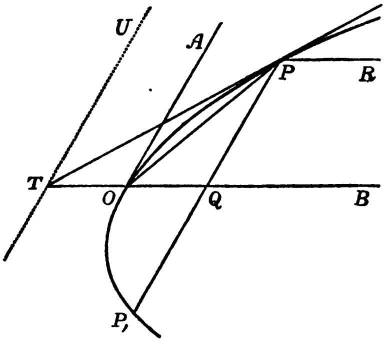
For any number of masses, expressed generally by m at the extremity of the vector α, we have the vector of the centre of inertia

$$\beta=\frac{\Sigma(m\alpha)}{\Sigma(m)}.$$

This may be written

$$\Sigma m(\alpha-\beta)=0.$$

Now $\alpha_1-\beta$ is the vector of m_1 with respect to the centre of inertia. Hence the theorem, *If the vector of each element of a mass, drawn from the centre of inertia, be increased in length in proportion to the mass of the element, the sum of all these vectors is zero.*



(*f.*) We see at once that the equation

$$\rho=\alpha t+\frac{\beta t^2}{2},$$

where t is an indeterminate number, and α, β given vectors, represents a parabola. The origin, O, is a point on the curve, β is parallel to the axis, i. e. is the diameter OB drawn from the origin, and α is OA the tangent at the origin. In the figure

$$\overline{QP}=\alpha t,\quad \overline{OQ}=\frac{\beta t^2}{2}.$$

The secant joining the points where t has the values t and t' is represented by the equation

$$\begin{aligned}\rho &= \alpha t+\frac{\beta t^2}{2}+x\left(\alpha t'+\frac{\beta t'^2}{2}-\alpha t-\frac{\beta t^2}{2}\right) \quad (\S\ 30)\\ &= \alpha t+\frac{\beta t^2}{2}+x(t'-t)\left\{\alpha+\beta\frac{t'+t}{2}\right\}.\end{aligned}$$

Put $t'=t$, and write x for $x(t'-t)$ [which may have any value] and the equation of the tangent at the point (t) is

$$\rho = \alpha t+\frac{\beta t^2}{2}+x(\alpha+\beta t).$$

Put $x=-t$, $\qquad \rho = -\frac{\beta t^2}{2}$,

or the intercept of the tangent on the diameter is — the abscissa of the point of contact.

Otherwise: the tangent is parallel to the vector $\alpha+\beta t$ or $\alpha t+\beta t^2$ or $\alpha t+\frac{\beta t^2}{2}+\frac{\beta t^2}{2}$ or $\overline{OQ}+\overline{OP}$. But $\overline{TP}=\overline{TO}+\overline{OP}$, hence $\overline{TO}=\overline{OQ}$.

(*g*.) Since the equation of any tangent to the parabola is

$$\rho = \alpha t+\frac{\beta t^2}{2}+x(\alpha+\beta t),$$

let us find the tangents which can be drawn from a given point. Let the vector of the point be

$$\rho = p\alpha+q\beta \quad (\S\ 24).$$

Since the tangent is to pass through this point, we have, as conditions to determine t and x,

$$t+x = p,$$

$$\frac{t^2}{2}+xt = q;$$

by equating respectively the coefficients of α and β.

Hence $\qquad t = p \pm\sqrt{p^2-2q}.$

Thus, in general, *two* tangents can be drawn from a given point. These coincide if

$$p^2 = 2q;$$

that is, if the vector of the point from which they are to be drawn is

$$\rho = p\alpha + q\beta = p\alpha + \frac{p^2}{2}\beta,$$

i. e. if the point lies *on* the parabola. They are imaginary if $2q > p^2$, i. e. if the point be

$$\rho = p\alpha + \left(\frac{p^2}{2} + r\right)\beta,$$

r being *positive*. Such a point is evidently *within* the curve, as at R, where $\overline{OQ} = \frac{p^2}{2}\beta$, $\overline{QP} = p\alpha$, $\overline{PR} = r\beta$.

(*h.*) Calling the values of t for the two tangents found in (*g*) t_1 and t_2 respectively, it is obvious that the vector joining the points of contact is

$$\alpha t_1 + \frac{\beta t_1^2}{2} - \alpha t_2 - \frac{\beta t_2^2}{2},$$

which is parallel to

$$\alpha + \beta\frac{t_1 + t_2}{2};$$

or, by the values of t_1 and t_2 in (*g*),

$$\alpha + p\beta.$$

Its direction, therefore, does not depend on q. In words, *If pairs of tangents be drawn to a parabola from points of a diameter produced, the chords of contact are parallel to the tangent at the vertex of the diameter.* This is also proved by a former result, for we must have $\overline{OT}$ for *each* tangent equal to $\overline{QO}$.

(*i.*) The equation of the chord of contact, for the point whose vector is

$$\rho = p\alpha + q\beta,$$

is thus

$$\rho = \alpha t_1 + \frac{\beta t_1^2}{2} + y(\alpha + p\beta).$$

Suppose this to pass always through the point whose vector is

$$\rho = a\alpha + b\beta.$$

Then we must have

$$\left.\begin{aligned} t_1 + y &= a, \\ \frac{t_1^2}{2} + py &= b; \end{aligned}\right\}$$

or

$$t_1 = p \pm \sqrt{p^2 - 2pa + 2b}.$$

Comparing this with the expression in (g), we have

$$q = pa - b;$$

that is, the point from which the tangents are drawn has the vector

$$\begin{aligned} \rho &= p\alpha + (pa - b)\beta \\ &= -b\beta + p(\alpha + a\beta), \text{ a straight line (§ 28 (1)).} \end{aligned}$$

The mere form of this expression contains the proof of the usual properties of the pole and polar in the parabola; but, for the sake of the beginner, we adopt a simpler, though equally general, process.

Suppose $a = 0$. This merely restricts the pole to the particular diameter to which we have referred the parabola. Then the pole is Q, where

$$\rho = b\beta;$$

and the polar is the line TU, for which

$$\rho = -b\beta + p\alpha.$$

Hence *the polar of any point is parallel to the tangent at the extremity of the diameter on which the point lies, and its intersection with that diameter is as far beyond the vertex as the pole is within, and vice versâ.*

(j.) As another example let us prove the following theorem. *If a triangle be inscribed in a parabola, the three points in which the sides are met by tangents at the angles lie in a straight line.*

Since O is any point of the curve, we may take it as one corner of the triangle. Let t and t_1 determine the others. Then, if ϖ_1, ϖ_2, ϖ_3 represent the vectors of the points of intersection of the tangents with the sides, we easily find

$$\varpi_1 = \frac{t_1^2}{2t_1 - t}\left(\alpha + \frac{t}{2}\beta\right),$$

$$\varpi_2 = \frac{t^2}{2t - t_1}\left(\alpha + \frac{t_1}{2}\beta\right),$$

$$\varpi_3 = \frac{tt_1}{t_1 + t}\alpha.$$

These values give

$$\frac{2t_1 - t}{t_1}\varpi_1 - \frac{2t - t_1}{t}\varpi_2 - \frac{t_1^2 - t^2}{tt_1}\varpi_3 = 0.$$

Also $$\frac{2t_1 - t}{t_1} - \frac{2t - t_1}{t} - \frac{t_1^2 - t^2}{tt_1} = 0 \text{ identically.}$$

Hence, by § 30, the proposition is proved.

(*k*.) Other interesting examples of this method of treating curves will, of course, suggest themselves to the student. Thus

$$\rho = \alpha \cos t + \beta \sin t$$

or $$\rho = \alpha x + \beta\sqrt{1 - x^2}$$

represents an ellipse, of which the given vectors α and β are semi-conjugate diameters.

Again,

$$\rho = \alpha t + \frac{\beta}{t} \quad \text{or} \quad \rho = \alpha \tan x + \beta \cot x$$

evidently represents a hyperbola referred to its asymptotes.

But, so far as we have yet gone, as we are not prepared to determine the lengths or inclinations of vectors, we can only investigate a very small class of the properties of curves, represented by such equations as those above written.

(*l*.) We may now, in extension of the statement in § 29, make the obvious remark that

$$\rho = \Sigma p\alpha$$

is the equation of a *curve* in space, if the numbers p_1, p_2, &c. are functions of *one* indeterminate. In such a case the equation is sometimes written

$$\rho = \phi(t).$$

But, if p_1, p_2, &c. be functions of *two* indeterminates, the locus of the extremity of ρ is a *surface;* whose equation is sometimes written

$$\rho = \phi(t, u).$$

(*m.*) Thus the equation

$$\rho = \alpha \cos t + \beta \sin t + \gamma t$$

belongs to a helix.

Again, $$\rho = p\alpha + q\beta + r\gamma$$

with a condition of the form

$$ap^2 + bq^2 + cr^2 = 1$$

belongs to a central surface of the second order, of which α, β, γ are the directions of conjugate diameters. If a, b, c be all positive, the surface is an ellipsoid.

32. In Example (*f*) above we performed an operation equivalent to the differentiation of a vector with reference to a single *numerical* variable of which it was given as an explicit function. As this process is of very great use, especially in quaternion investigations connected with the motion of a particle or point; and as it will afford us an opportunity of making a preliminary step towards overcoming the novel difficulties which arise in quaternion differentiation; we will devote a few sections to a more careful exposition of it.

33. It is a striking circumstance, when we consider the way in which Newton's original methods in the Differential Calculus have been decried, to find that Hamilton was *obliged* to employ them, and not the more modern forms, in order to overcome the characteristic difficulties of quaternion differentiation. Such a thing as *a differential coefficient has absolutely no meaning in* *quaternions*, except in those special cases in which we are dealing with degraded quaternions, such as numbers, Cartesian coördinates, &c. But a quaternion expression has always a *differential*, which is, simply, what Newton called a *fluxion*.

As with the Laws of Motion, the basis of Dynamics, so with the foundations of the Differential Calculus; we are gradually coming to the conclusion that Newton's system is the best after all.

34. Suppose ρ to be the vector of a curve in space. Then, generally, ρ may be expressed as the sum of a number of terms, each of which is a multiple of a given vector by a function of some *one* indeterminate; or, as in § 31 (l), if P be a point on the curve,

$$\overline{OP} = \rho = \phi(t).$$

And, similarly, if Q be any other point on the curve,

$$\overline{OQ} = \rho_1 = \phi(t_1) = \phi(t + \delta t),$$

where δt is *any number whatever*.

The vector-chord $\overline{PQ}$ is therefore, rigorously,

$$\delta\rho = \rho_1 - \rho = \phi(t + \delta t) - \phi t.$$

35. It is obvious that, in the present case, *because the vectors involved in ϕ are constant, and their numerical multipliers alone vary*, the expression $\phi(t + \delta t)$ is, by Taylor's Theorem, equivalent to

$$\phi(t) + \frac{d\phi(t)}{dt}\delta t + \frac{d^2\phi(t)}{dt^2}\frac{\delta t^2}{1.2} + \ldots\ldots$$

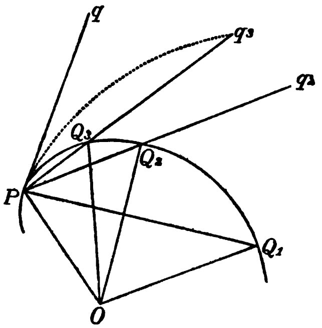
Hence,

$$\delta\rho = \frac{d\phi(t)}{dt}\delta t + \frac{d^2\phi(t)}{dt^2}\frac{\delta t^2}{1.2} + \&c.$$

And we are thus entitled to write, when δt has been made indefinitely small,

$$\text{Limit}\left(\frac{\delta\rho}{\delta t}\right)_{\delta t=0} = \frac{d\rho}{dt} = \frac{d\phi(t)}{dt} = \phi'(t).$$

In such a case as this, then, we are permitted to differentiate, or to form the differential coefficient of, a vector according to the ordinary rules of the Differential Calculus. But great additional insight into the process is gained by applying Newton's method.



36. Let $\overline{OP}$ be

$$\rho = \phi(t),$$

and $\overline{OQ_1}$

$$\rho = \phi(t+dt),$$

where *dt* is any number whatever.

The number *t* may here be taken as representing *time*, i.e. we may suppose a point to move along the curve in such a way that the value of *t* for the vector of point *P* of the curve denotes the interval which has elapsed (since a fixed epoch) when the moving point has reached the extremity of that vector. If, then, *dt* represent any interval, finite or not, we see that

$$\overline{OQ_1} = \phi(t+dt)$$

will be the vector of the point after the additional interval *dt*.

But this, in general, gives us little or no information as to the velocity of the point at *P*. We shall get a better approximation by halving the interval *dt*, and finding Q_2, where $OQ_2 = \phi(t+\frac{1}{2}dt)$, as the position of the moving point at that time. Here the vector virtually described in $\frac{1}{2}dt$ is $\overline{PQ_2}$. To find, on this supposition, the vector described in *dt*, we must double this, and we find, as a second approximation to the vector which the moving point would have described in time *dt*, if it had moved for that period in the direction and with the velocity it had at *P*,

$$\overline{Pq_2} = 2\overline{PQ_2} = 2(\overline{OQ_2} - \overline{OP})$$

$$= 2\{\phi(t+\tfrac{1}{2}dt) - \phi t\}.$$

The next approximation gives

$$\overline{Pq}_3 = 3\overline{PQ}_3 = 3\,(\overline{OQ}_3 - \overline{OP})$$
$$= 3\,\{\phi(t+\tfrac{1}{3}dt) - \phi t\}.$$

And so on, each step evidently leading us nearer the sought truth. Hence, to find the vector which would have been described in time *dt* had the circumstances of the motion at *P* remained undisturbed, we must find the value of

$$d\rho = \overline{Pq} = \mathcal{L}_{x=\infty}\, x\left\{\phi\left(t+\frac{1}{x}dt\right) - \phi t\right\}.$$

We have seen that in this particular case we may use Taylor's Theorem. We have, therefore,

$$d\rho = \mathcal{L}_{x=\infty}\, x\left\{\phi'(t)\frac{1}{x}dt + \phi''(t)\frac{1}{x^2}\frac{(dt)^2}{1.2} + \&\text{c.}\right\}$$
$$= \phi'(t)\,dt.$$

And, if we choose, we may now write

$$\frac{d\rho}{dt} = \phi'(t).$$

37. But it is to be most particularly remarked that in the whole of this investigation no regard whatever has been paid to the magnitude of *dt*. The question which we have now answered may be put in the form—*A point describes a given curve in a given manner. At any point of its path its motion suddenly ceases to be accelerated. What space will it describe in a definite interval?* As Hamilton well observes, this is, for a planet or comet, the case of a "celestial Attwood's machine."

38. If we suppose the variable, in terms of which ρ is expressed, to be the arc, s, of the curve measured from some fixed point, we find as before

$$d\rho = \phi'(t)\,dt = \frac{\phi'(t)\,ds}{\frac{ds}{dt}}$$
$$= \phi'(s)\,ds.$$

From the very nature of the question it is obvious that the length of $d\rho$ must in this case be ds. This remark is of importance, as we shall see later; and it may therefore be useful to obtain afresh the above result without any reference to time or velocity.

39. Following strictly the process of Newton's VIIth Lemma, let us describe on Pq_2 an arc similar to PQ_2, and so on. Then obviously, as the subdivision of ds is carried farther, the new arc (whose length is always ds) more and more nearly coincides with the line which expresses the corresponding approximation to $d\rho$.

40. As a final example let us take the hyperbola

$$\rho = \alpha t + \frac{\beta}{t}.$$

Here

$$d\rho = \left(\alpha - \frac{\beta}{t^2}\right) dt.$$

This shews that the tangent is parallel to the vector

$$\alpha t - \frac{\beta}{t}.$$

In words, *if the vector (from the centre) of a point in a hyperbola be one diagonal of a parallelogram, two of whose sides coincide with the asymptotes, the other diagonal is parallel to the tangent at the point.*

41. Let us reverse this question, and *seek the envelop of a line which cuts off from two fixed axes a triangle of constant area.*

If the axes be in the directions of α and β, the intercepts may evidently be written αt and $\frac{\beta}{t}$. Hence the equation of the line is (§ 30)

$$\rho = \alpha t + x\left(\frac{\beta}{t} - \alpha t\right).$$

The condition of envelopment is, obviously,

$$d\rho = 0.$$

This gives

$$0 = \left\{\alpha - x\left(\frac{\beta}{t^2} + \alpha\right)\right\} dt + \left(\frac{\beta}{t} - \alpha t\right) dx.$$

[*We are not here to equate to zero the coefficients of dt* and *dx;* for we must remember that this equation is of the form

$$0 = p\alpha + q\beta,$$

where p and q are numbers; and that, so long as α and β are actual and non-parallel vectors, the existence of such an equation requires $$p = 0, \quad q = 0.]$$

Hence $$(1-x)dt - tdx = 0,$$

and $$-\frac{x}{t^2}dt + \frac{dx}{t} = 0.$$

From these, at once, $x = \frac{1}{2}$, since dx and dt are indeterminate. Thus the equation of the envelop is

$$\rho = \alpha t + \tfrac{1}{2}\left(\frac{\beta}{t} - \alpha t\right)$$

$$= \tfrac{1}{2}\left(\alpha t + \frac{\beta}{t}\right),$$

the hyperbola as before; α, β being portions of its asymptotes.

42. It may assist the student to a thorough comprehension of the above process, if we put it in a slightly different form. Thus the equation of the enveloping line may be written

$$\rho = \alpha t(1-x) + \beta\frac{x}{t},$$

which gives $d\rho = 0 = \alpha d(t(1-x)) + \beta d\left(\frac{x}{t}\right)$.

Hence, as α is not parallel to β, we must have

$$d(t(1-x)) = 0, \qquad d\left(\frac{x}{t}\right) = 0;$$

and these are, when expanded, the equations we obtained in the preceding section.

43. For farther illustration we give a solution not directly employing the differential calculus. The equations of any two of the enveloping lines are

$$\rho = at + x\left(\frac{\beta}{t} - at\right),$$

$$\rho = at_1 + y\left(\frac{\beta}{t_1} - at_1\right),$$

t and t_1 being given, while x and y are indeterminate.

At the point of intersection of these lines we have (§ 26),

$$\left.\begin{aligned} t(1-x) &= t_1(1-y), \\ \frac{x}{t} &= \frac{y}{t_1}. \end{aligned}\right\}$$

These give, by eliminating y,

$$t(1-x) = t_1\left(1 - \frac{t_1}{t}x\right),$$

or $$x = \frac{t}{t_1+t}.$$

Hence the vector of the point of intersection is

$$\rho = \frac{att_1+\beta}{t_1+t},$$

and thus, for the ultimate intersections, where $\mathcal{L}\frac{t_1}{t} = 1$,

$$\rho = \tfrac{1}{2}\left(at + \frac{\beta}{t}\right) \text{ as before.}$$

COR. (1). If $tt_1 = 1$,

$$\rho = \frac{a+\beta}{t+\frac{1}{t}};$$

or the intersection lies in the diagonal of the parallelogram on a, β.

COR. (2). If $t_1 = mt$, where m is constant,

$$\rho = \frac{mta + \frac{\beta}{t}}{m+1}.$$

But we have also $x = \frac{1}{m+1}$.

Hence *the locus of a point which divides in a given ratio a line cutting off a given area from two fixed axes, is a hyperbola of which these axes are the asymptotes.*

COR. (3). If we take

$$tt_1(t+t_1) = \text{constant}$$

the locus is a parabola; and so on.

44. The reader who is fond of Anharmonic Ratios and Transversals will find in the early chapters of Hamilton's *Elements of Quaternions* an admirable application of the composition of vectors to these subjects. The Theory of Geometrical Nets, in a plane, and in space, is there very fully developed; and the method is shewn to include, as particular cases, the processes of Grassmann's *Ausdehnungslehre* and Möbius' *Barycentrische Calcul.* Some very curious investigations connected with curves and surfaces of the second and third orders are also there founded upon the composition of vectors.

EXAMPLES TO CHAPTER I.

1. The lines which join, towards the same parts, the extremities of two equal and parallel lines are themselves equal and parallel. (*Euclid,* I. xxxiii.)

2. Find the vector of the middle point of the line which joins the middle points of the diagonals of any quadrilateral, plane or gauche, the vectors of the corners being given; and so prove that this point is the mean point of the quadrilateral.

If two opposite sides be divided proportionally, and two new quadrilaterals be formed by joining the points of division, the mean points of the three quadrilaterals lie in a straight line.

Shew that the mean point may also be found by bisecting the line joining the middle points of a pair of opposite sides.

3. Verify that the property of the coefficients of three vectors whose extremities are in a line (§ 30) is not interfered with by altering the origin.

4. If two triangles ABC, abc, be so situated in space that Aa, Bb, Cc meet in a point, the intersections of AB, ab, of BC, bc, and of CA, ca, lie in a straight line.

5. Prove the converse of 4, i. e. if lines be drawn, one in each of two planes, from any three points in the straight line in which these planes meet, the two triangles thus formed are sections of a common pyramid.

6. If five quadrilaterals be formed by omitting in succession each of the sides of any pentagon, the lines bisecting the diagonals of these quadrilaterals meet in a point. (H. Fox Talbot.)

7. Assuming, as in § 7, that the operator

$$\cos\theta + \sqrt{-1}\sin\theta$$

turns any radius of a given circle through an angle θ in the positive direction of rotation, without altering its length, deduce the ordinary formulæ for $\cos(A+B)$, $\cos(A-B)$, $\sin(A+B)$, and $\sin(A-B)$, in terms of sines and cosines of A and B.

8. If two tangents be drawn to a hyperbola, the line joining the centre with their point of intersection bisects the lines joining the points where the tangents meet the asymptotes: and the tangent at the point where it meets the curves bisects the intercepts of the asymptotes.

9. Any two tangents, limited by the asymptotes, divide each other proportionally.

10. If a chord of a hyperbola be one diagonal of a parallelogram whose sides are parallel to the asymptotes, the other diagonal passes through the centre.

11. Show that

$$\rho = x^2 \alpha + y^2 \beta + (x+y)^2 \gamma$$

is the equation of a cone of the second degree, and that its section by the plane

$$\rho = \frac{p\alpha + q\beta + r\gamma}{p+q+r}$$

is an ellipse which touches, at their middle points, the sides of the triangle of whose corners α, β, γ are the vectors. (Hamilton, *Elements*, p. 96.)

12. The lines which divide, proportionally, the pairs of opposite sides of a gauche quadrilateral, are the generating lines of a hyperbolic paraboloid. (*Ibid.* p. 97.)

13. Show that

$$\rho = x^3 \alpha + y^3 \beta + z^3 \gamma,$$

where

$$x + y + z = 0,$$

represents a cone of the third order, and that its section by the plane

$$\rho = \frac{p\alpha + q\beta + r\gamma}{p+q+r}$$

is a cubic curve, of which the lines

$$\rho = \frac{p\alpha + q\beta}{p+q}, \text{ \&c.}$$

are the asymptotes and the three (real) tangents of inflexion. Also that the mean point of the triangle formed by these lines is a conjugate point of the curve. Hence that the vector $\alpha + \beta + \gamma$ is a conjugate ray of the cone. (*Ibid.* p. 96.)

CHAPTER II.

PRODUCTS AND QUOTIENTS OF VECTORS.

45. WE now come to the consideration of points in which the Calculus of Quaternions differs entirely from any previous mathematical method; and here we shall get an idea of what a Quaternion is, and whence it derives its name. These points are fundamentally involved in the novel use of the symbols of multiplication and division. And the simplest introduction to the subject seems to be the consideration of the quotient, or ratio, of two vectors.

46. If the given vectors be parallel to each other, we have already seen (§ 22) that either may be expressed as a *numerical* multiple of the other; the multiplier being simply the ratio of their lengths, taken positively if they are similarly directed, negatively if they run opposite ways.

47. If they be not parallel, let $\overline{OA}$ and $\overline{OB}$ be drawn parallel and equal to them from any point O; and the question is reduced to finding the value of the ratio of two vectors drawn from the same point. Let us try to find *upon how many distinct numbers this ratio depends.*

We may suppose $\overline{OA}$ to be changed into $\overline{OB}$ by the following processes.

1st. Increase or diminish the length of $\overline{OA}$ till it becomes equal to that of $\overline{OB}$. For this only *one* number is required, viz. the ratio of the lengths of the two vectors. As Hamilton remarks, this is a positive, or rather a *signless*, number.

2nd. Turn $\overline{OA}$ about O until its direction coincides with that of $\overline{OB}$, and (remembering the effect of the first

> operation) we see that the two vectors now coincide or become identical. To specify this operation *three* more numbers are required, viz. *two* angles (such as node and inclination in the case of a planet's orbit) to fix the plane in which the rotation takes place; and *one* angle for the amount of this rotation.

Thus it appears that the ratio of two vectors, or the multiplier required to change one vector into another, in general depends upon *four* distinct numbers, whence the name QUATERNION.

48. It is obvious that the operations just described may be performed, with the same result, in the opposite order, being perfectly independent of each other. Thus it appears that a quaternion, considered as the factor or agent which changes one definite vector into another, may itself be decomposed into two factors of which the order is immaterial.

The *stretching* factor, or that which performs the first operation in § 47, is called the TENSOR, and is denoted by prefixing T to the quaternion considered.

The *turning* factor, or that corresponding to the second operation in § 47, is called the VERSOR, and is denoted by the letter U prefixed to the quaternion.

49. Thus, if $\overline{OA} = \alpha$, $\overline{OB} = \beta$, and if q be the quaternion which changes α to β, we have

$$\beta = q\alpha,$$

which we may write in the form

$$\frac{\beta}{\alpha} = q, \quad \text{or} \quad \beta\alpha^{-1} = q,$$

if we agree to *define* that

$$\frac{\beta}{\alpha}.\alpha = \beta\alpha^{-1}.\alpha = \beta.$$

Here it is to be particularly noticed that we write q *before* α to signify that α is multiplied by q, not q multiplied by α.

[This remark is of extreme importance in quaternions, for, as we shall soon see, the Commutative Law does not generally apply to the factors of a product.]

We have also, by § 47,

$$q = Tq\,Uq = Uq\,Tq,$$

where, as before, Tq depends merely on the relative lengths of α and β, and Uq depends solely on their directions.

Thus, if α_1 and β_1 be vectors of unit length parallel to α and β respectively,

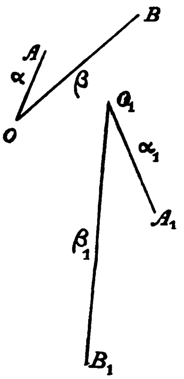
$$T\frac{\beta_1}{\alpha_1} = 1, \quad U\frac{\beta_1}{\alpha_1} = U\frac{\beta}{\alpha}.$$

50. We must now carefully notice that the quaternion which is the quotient when β is divided by α in no way depends upon the *absolute* lengths, or directions, of these vectors. Its value will remain unchanged if we substitute for them any other pair of vectors which

(1) have their lengths in the same ratio,

(2) have their common plane the same or parallel,

and (3) make the same angle with each other.



Thus in the annexed figure

$$\frac{\overline{O_1B_1}}{\overline{O_1A_1}} = \frac{\overline{OB}}{\overline{OA}}$$

if, and only if,

(1) $\frac{O_1B_1}{O_1A_1} = \frac{OB}{OA}$,

(2) plane AOB parallel to plane $A_1O_1B_1$,

(3) $\angle AOB = \angle A_1O_1B_1$.

[Equality of angles is understood to include similarity in direction. Thus the rotation about an upward axis is negative (or right-handed) from OA to OB, and also from O_1A_1 to O_1B_1.]

51. The *Reciprocal* of a quaternion q is defined by the equation,

$$q\frac{1}{q} = qq^{-1} = 1.$$

Hence if $$\frac{\beta}{\alpha} = q, \text{ or}$$

$$\beta = q\alpha,$$

we must have $$\frac{\alpha}{\beta} = \frac{1}{q} = q^{-1}.$$

For this gives $$\frac{\alpha}{\beta}.\beta = q^{-1}.q\alpha,$$

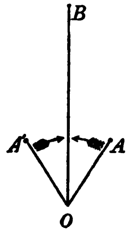
and each side is evidently equal to α.

Or, we may reason thus, q changes $\overline{OA}$ to $\overline{OB}$, q^{-1} must therefore change $\overline{OB}$ to $\overline{OA}$, and is therefore expressed by $\frac{\alpha}{\beta}$ (§ 49).

The tensor of the reciprocal of a quaternion is therefore the reciprocal of the tensor; and the versor differs merely by the *reversal* of its representative angle.

52. The *Conjugate* of a quaternion q, written Kq, has the same tensor, plane, and angle, only the angle is taken the reverse way. Thus, if

$$OA' = OA, \text{ and } \angle A'OB = \angle AOB,$$



$$\frac{\overline{OB}}{\overline{OA}} = q, \qquad \frac{\overline{OB}}{\overline{OA'}} = \text{conjugate of } q = Kq.$$

By last section we see that

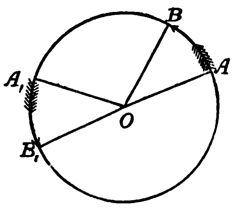
$$Kq = (Tq)^2 q^{-1},$$

Hence $$qKq = Kq.q = (Tq)^2.$$

This proposition is obvious, if we recollect that the tensors of q and Kq are equal, and that the versors are such that either *reverses* the effect of the other. The joint effect of these factors is therefore merely to multiply twice over by the common tensor.

53. It is evident from the results of § 50 that, if α and β

be of equal length, their quaternion quotient becomes a versor (the tensor being unity) and may be represented indifferently by any one of an infinite number of arcs of given length lying on the circumference of a circle, of which the two vectors are radii. This is of considerable importance in the proofs which follow.



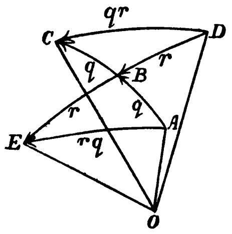
Thus the versor $\frac{\overline{OB}}{\overline{OA}}$ may be represented by the arc AB, which may be written $\overset{\frown}{AB}$.

And, similarly, the versor $\frac{\overline{OB_1}}{\overline{OA_1}}$ is represented by $\overset{\frown}{A_1B_1}$ which is *equal* to (and measured in the same direction as) $\overset{\frown}{AB}$ if

$$\angle A_1OB_1 = \angle AOB,$$

i. e. if the versors are equal.

54. By the aid of this process, when a versor is represented by an arc of a great circle on the unit-sphere, we can easily prove that *quaternion multiplication is not generally commutative.*



Thus let q be the versor $\overset{\frown}{AB}$ or $\frac{\overline{OB}}{\overline{OA}}$. Make $\overset{\frown}{BC} = \overset{\frown}{AB}$, then q may also be represented by $\frac{\overline{OC}}{\overline{OB}}$.

In the same way any other versor r may be represented by $\overset{\frown}{DB}$ or $\overset{\frown}{BE}$ and by $\frac{\overline{OB}}{\overline{OD}}$ or $\frac{\overline{OE}}{\overline{OB}}$.

The line OB in the figure is definite, and is given by the intersection of the planes of the two versors; O being the centre of the unit-sphere.

Now $\quad r\overline{OD} = \overline{OB}$, and $q\overline{OB} = \overline{OC}$,

Hence $\quad qr\overline{OD} = \overline{OC}$, or $qr = \frac{\overline{OC}}{\overline{OD}}$, and may therefore be represented by the arc $\overset{\frown}{DC}$ of a great circle.

But rq is easily seen to be represented by the arc $\overset{\frown}{AE}$.

For $\quad q\overline{OA} = \overline{OB}, \quad$ and $\quad r\overline{OB} = \overline{OE},$

whence $\quad rq\overline{OA} = \overline{OE}, \quad$ and $\quad rq = \frac{\overline{OE}}{\overline{OA}}.$

Thus the versors rq and qr, though represented by arcs of equal length, are not generally in the same plane and are therefore unequal: unless the planes of q and r coincide.

Calling $\overline{OA}$ a, we see that we have assumed, or defined, in the above proof, that $q.ra = qr.a$ and $r.qa = rq.a$ when qa, ra, qra, and rqa are all *vectors*.

55. Obviously $\overset{\frown}{CB}$ is Kq, $\overset{\frown}{BD}$ is Kr, and $\overset{\frown}{CD}$ is $K(qr)$. But $\overset{\frown}{CD} = \overset{\frown}{BD}.\overset{\frown}{CB}$, which gives us the very important theorem

$$K(qr) = Kr.Kq.$$

56. The propositions just proved are, of course, true of quaternions as well as of versors; for the former involve only an additional numerical factor which has reference to the length merely, and not the direction, of a vector (§ 48).

57. Seeing thus that the commutative law does not in general hold in the multiplication of quaternions, let us enquire whether the Associative Law holds. That is, if p, q, r be three quaternions, have we

$$p.qr = pq.r?$$

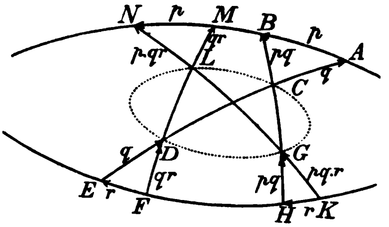
This is, of course, obviously true if p, q, r be numerical quantities, or even any of the imaginaries of algebra. But it cannot be considered as a truism for symbols which do not in general give

$$pq = qp.$$

58. In the first place we remark that p, q, and r may be considered as versors only, and therefore represented by arcs of great circles, for their tensors may obviously (§ 48) be divided out from both sides, being commutative with the versors.

Let $\widehat{AB} = p$, $\widehat{ED} = \widehat{CA} = q$, and $\widehat{FE} = r$.

Join BC and produce the great circle till it meets EF in H, and make $\widehat{KH} = \widehat{FE} = r$, and $\widehat{HG} = \widehat{CB} = pq$ (§ 54).



Join GK. Then $\widehat{KG} = \widehat{HG}.\widehat{KH} = pq.r$.

Join FD and produce it to meet AB in M. Make

$$\widehat{LM} = \widehat{FD},$$

and $\widehat{MN} = \widehat{AB}$, and join NL. Then

$$\widehat{LN} = \widehat{MN}.\widehat{LM} = p.qr.$$

Hence to shew that $p.qr = pq.r$ all that is requisite is to prove that LN, and KG, described as above, are *equal arcs of the same great circle*, since, by the figure, they are evidently measured in the same direction. This is perhaps most easily effected by the help of the fundamental properties of the curves known as *Spherical Conics*. As they are not usually familiar to students, we make a slight digression for the purpose of proving these fundamental properties; after Chasles, by whom and Magnus they were discovered. An independent proof of the associative principle will presently be indicated, and in Chapter VII we shall employ quaternions to give an independent proof of the theorems now to be established.

59.* DEF. *A spherical conic is the curve of intersection of a cone of the second degree with a sphere, the vertex of the cone being the centre of the sphere.*

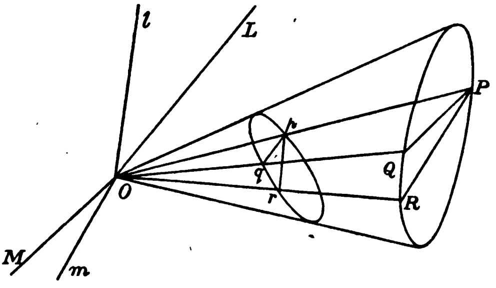
If a cone have one series of circular sections, it has another series, and any two circles belonging to different series lie on a sphere. This is easily proved as follows.

Describe a sphere, A, cutting the cone in one circular section,

C, and in any other point whatever, and let the side OpP of the cone meet A in p, P; P being a point in C. Then $PO\cdot Op$ is constant, and, therefore, since P lies in a plane, p lies on a sphere, a, passing through O. Hence the locus, c, of p is a circle, being the intersection of the two spheres A and a.

Let OqQ be any other side of the cone, q and Q being points in c, C, respectively. Then the quadrilateral $qQPp$ is inscribed in a circle (that in which its plane cuts the sphere A) and the exterior angle at p is equal to the interior angle at Q. If OL, OM be the lines in which the plane POQ cuts the *cyclic planes* (planes through O parallel to the two series of circular sections) they are obviously parallel to pq, QP, respectively; and therefore

$$\angle LOp = \angle Opq = \angle OQP = \angle MOQ.$$

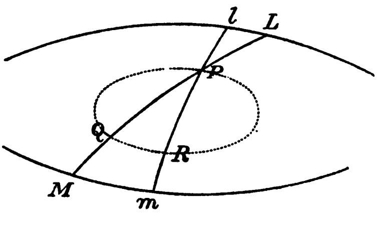


Let any third side, OrR, of the cone be drawn, and let the plane OPR cut the cyclic planes in Ol, Om respectively. Then, evidently,

$$\angle lOL = \angle qpr, \quad \angle MOm = \angle QPR,$$

and these angles are independent of the position of the points p and P, if Q and R be fixed points.

In a section of the above diagram by a sphere whose centre is O, lL, Mm are the great circles which represent the cyclic planes, PQR is the spherical conic which represents the cone. The point P represents the line OpP, and so with the others.



The propositions above may now be stated thus

$$\text{Arc}\, PL = \text{arc}\, MQ\,;$$

and, if Q and R be fixed, Mm and lL are constant arcs whatever be the position of P.

60. The application to § 58 is now obvious. In the figure of that article we have

$$\widehat{FE} = \widehat{KH},\ \widehat{ED} = \widehat{CA},\ \widehat{HG} = \widehat{CB},\ \widehat{LM} = \widehat{FD}.$$

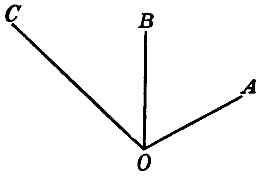
Hence L, C, G, D are points of a spherical conic whose cyclic planes are those of AB, FE. Hence also KG passes through L, and with LM intercepts on AB an arc equal to $\widehat{AB}$. That is, it passes through N, or KG and LN are arcs of the same great circle: and they are equal, for G and L are points in the spherical conic.

Also, the associative principle holds for any number of quaternion factors. For, obviously,

$$qr.st = qrs.t, \text{ \&c., \&c.,}$$

since we may consider qr as a single quaternion, and the above proof applies directly.

61. That quaternion addition, and therefore also subtraction, is commutative, it is easy to shew.



For if the planes of two quaternions, q and r, intersect in the line OA, we may take any vector $\overline{OA}$ in that line, and at once find two others, $\overline{OB}$ and $\overline{OC}$, such that

$$\overline{OB} = q\overline{OA},$$

and $$\overline{OC} = r\overline{OA}.$$

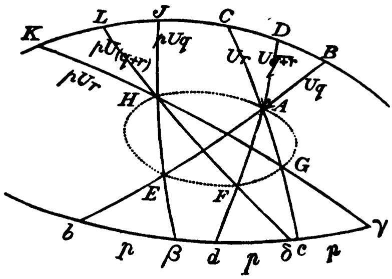
And $$(q+r)\overline{OA} = \overline{OB} + \overline{OC} = \overline{OC} + \overline{OB} = (r+q)\overline{OA},$$

since *vector* addition is commutative (§ 27).

Here it is obvious that $(q+r)\,\overline{OA}$, being the diagonal of the parallelogram on $\overline{OB}$, $\overline{OC}$, divides the angle between OB and OC

in a ratio depending solely on the ratio of the lengths of these lines, i. e. on the ratio of the tensors of q and r. This will be useful to us in the proof of the distributive law, to which we proceed.

62. Quaternion multiplication, and therefore division, is distributive. One simple proof of this depends on the possibility, shortly to be proved, of representing *any* quaternion as a linear function of three given rectangular unit-vectors. And when the proposition is thus established, the associative principle may readily be deduced from it.



But we may employ for its proof the properties of Spherical Conics already employed in demonstrating the truth of the associative principle. For continuity we give an outline of the proof by this process.

Let $\overset{\frown}{BA}$, $\overset{\frown}{CA}$ represent the versors of q and r, and bc the great circle whose plane is that of p.

Then, if we take as operand the vector $\overline{OA}$, it is obvious that $U(q+r)$ will be represented by some such arc as $\overset{\frown}{DA}$ where B, D, C are in one great circle; for $(q+r)\overline{OA}$ is in the same plane as $q\overline{OA}$ and $r\overline{OA}$, and the relative magnitudes of the arcs BD and DC depend solely on the tensors of q and r. Produce BA, DA, CA to meet bc in b, d, c respectively, and make

$$\overset{\frown}{Eb} = \overset{\frown}{BA}, \quad \overset{\frown}{Fd} = \overset{\frown}{DA}, \quad \overset{\frown}{Gc} = \overset{\frown}{CA}.$$

Also make $\overset{\frown}{b\beta} = \overset{\frown}{d\delta} = \overset{\frown}{c\gamma} = p$. Then E, F, G, A lie on a spherical conic of which BC and bc are the cyclic arcs. And, because $\overset{\frown}{b\beta} = \overset{\frown}{d\delta} = \overset{\frown}{c\gamma}$, $\overset{\frown}{\beta E}$, $\overset{\frown}{\delta F}$, $\overset{\frown}{\gamma G}$, when produced, meet in a point H

which is also on the spherical conic (§ 59*). Let these arcs meet BC in J, L, K respectively. Then we have

$$\begin{aligned}\frown{JH} &= \frown{E\beta} = p\,Uq,\\ \frown{LH} &= \frown{F\delta} = p\,U(q+r),\\ \frown{KH} &= \frown{G\gamma} = p\,Ur.\end{aligned}$$

Also $$\frown{LJ} = \frown{BD},$$

and $$\frown{KL} = \frown{CD}.$$

And, on comparing the portions of the figure bounded respectively by HKJ and by ACB we see that (when considered with reference to their effects as factors multiplying $\overline{OH}$ and $\overline{OA}$ respectively)

$pU(q+r)$ bears the same relation to pUq and pUr

that $U(q+r)$ bears to Uq and Ur.

But $$T(q+r)\,U(q+r) = q+r = Tq\,Uq + Tr\,Ur.$$

Hence $$T(q+r).pU(q+r) = Tq.pUq + Tr.pUr;$$

or, since the tensors are mere numbers and commutative with all other factors,

$$p(q+r) = pq+pr.$$

In a similar manner it may be proved that

$$(q+r)p = qp+rp.$$

And then it follows at once that

$$(p+q)(r+s) = pr+ps+qr+qs.$$

63. By similar processes to those of § 53 we see that versors, and therefore also quaternions, are subject to the index-law

$$q^m.q^n = q^{m+n},$$

at least so long as m and n are positive integers.

The extension of this property to negative and fractional exponents must be deferred until we have defined a negative or fractional power of a quaternion.

64. We now proceed to the special case of *quadrantal* versors, from whose properties it is easy to deduce all the foregoing results of this chapter. These properties were indeed those whose discovery by Hamilton in 1843 led almost intuitively to the establishment of the Quaternion Calculus. We shall content ourselves at present with an assumption, which will be shewn to lead to consistent results; but at the end of the chapter we shall shew that no other assumption is possible, following for this purpose a very curious quasi-metaphysical speculation of Hamilton's.

65. Suppose we have a system of three mutually perpendicular unit-vectors, drawn from one point, which we may call for shortness I, J, K. Suppose also that these are so situated that a positive (i. e. *left-handed*) rotation through a right angle about I as an axis brings J to coincide with K. Then it is obvious that positive quadrantal rotation about J will make K coincide with I; and, about K, will make I coincide with J.

For definiteness we may suppose I to be drawn *eastwards*, J *northwards*, and K *upwards*. Then it is obvious that a positive (left-handed) rotation about the eastward line (I) brings the northward line (J) into a vertically upward position (K); and so of the others.

66. Now the operator which turns J into K is a quadrantal versor (§ 53); and, as its axis is the vector I, we may call it i.

Thus $$\frac{K}{J} = i, \quad \text{or} \quad K = iJ. \qquad (1)$$

Similarly we may put

$$\frac{I}{K} = j, \quad \text{or} \quad I = jK, \qquad (2)$$

and $$\frac{J}{I} = k, \quad \text{or} \quad J = kI. \qquad (3)$$

[It may here be noticed, merely to shew the symmetry of the system we are explaining, that if the three mutually perpendicular vectors I, J, K be made to revolve about a line equally inclined to all, so that I is brought to coincide with J, J will then coincide with K, and K with I: and the above equations will still hold good, only (1) will become (2), (2) will become (3), and (3) will become (1).]

67. By the results of § 50 we see that

$$\frac{-J}{K} = \frac{K}{J};$$

i. e. a southward unit-vector bears the same ratio to an upward unit-vector that the latter does to a northward one; and therefore we have

$$\frac{-J}{K} = i, \quad \text{or} \quad -J = iK. \quad \text{.....................} \quad (4)$$

Similarly $$\frac{-K}{I} = j, \quad \text{or} \quad -K = jI; \quad \text{.....................} \quad (5)$$

and $$\frac{-I}{J} = k, \quad \text{or} \quad -I = kJ. \quad \text{.....................} \quad (6)$$

68. By (4) and (1) we have

$$-J = iK = i(iJ) = i^2 J.$$

Hence $$i^2 = -1. \quad \text{..............................} \quad (7)$$

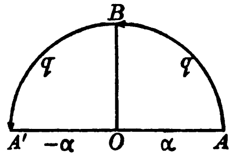
And, in the same way, (5) and (2) give

$$j^2 = -1, \quad \text{..............................} \quad (8)$$

and (6) and (3) $$k^2 = -1. \quad \text{..............................} \quad (9)$$

Thus, as the directions of I, J, K are perfectly arbitrary, we see that *the square of every quadrantal versor is negative unity.*

[Though the following proof is in principle exactly the same as the foregoing, it may perhaps be of use to the student, in shewing him precisely the nature as well as the simplicity of the step we have taken.



Let ABA' be a semicircle, whose centre is O, and let OB be perpendicular to AOA'.

Then $\frac{\overline{OB}}{\overline{OA}}$, $= q$ suppose, is a quadrantal versor, and is evidently equal to $\frac{\overline{OA'}}{\overline{OB}}$; §§ 50, 53.

Hence $$q^2 = \frac{\overline{OA'}}{\overline{OB}} \cdot \frac{\overline{OB}}{\overline{OA}} = \frac{\overline{OA'}}{\overline{OA}} = -1.]$$

69. Having thus found that the squares of i, j, k are each equal to negative unity; it only remains that we find the values of their products two and two. For, as we shall see, the result is such as to shew that the value of any other combination whatever of i, j, k (as factors of a product) may be deduced from the values of these squares and products.

Now it is obvious that

$$\frac{K}{-I} = \frac{I}{K} = j;$$

(i. e. the versor which turns a westward unit-vector into an upward one will turn the upward into an eastward unit);

$$\text{or} \quad K = j(-I) = -jI. \qquad (10)$$

[The negative sign, being a mere numerical factor, is evidently commutative with j; indeed we may, if necessary, easily assure ourselves of the fact that to turn the negative (or reverse) of a vector through a right (or indeed any) angle, is the same thing as to turn the vector through that angle and then reverse it.]

Now let us operate on the two equal vectors in (10) by the same versor, i, and we have

$$iK = i(-jI) = -ijI.$$

But by (4) and (3)

$$iK = -J = -kI.$$

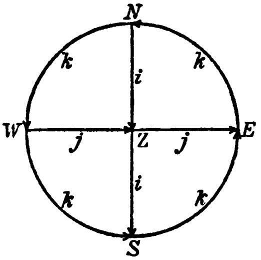
Comparing these equations we have

$$-ijI = -kI;$$

or, by § 54 (end), and symmetry gives

$$\left.\begin{aligned} ij &= k, \\ jk &= i, \\ ki &= j. \end{aligned}\right\} \quad \text{.............................(11)}$$

The meaning of these important equations is very simple; and is, in fact, obvious from our construction in § 54 for the multiplication of versors; as we see by the annexed figure, where we must remember that i, j, k are quadrantal versors whose planes are at right angles, so that the figure represents a hemisphere divided into quadrantal triangles.



Thus, to show that $ij = k$, we have, O being the centre of the sphere, N, E, S, W the north, east, south, and west, and Z the zenith (as in § 65);

$$j\,\overline{OW} = \overline{OZ},$$

whence

$$ij\,\overline{OW} = i\,\overline{OZ} = \overline{OS} = k\,\overline{OW}.$$

70. But, by the same figure,

$$i\,\overline{ON} = \overline{OZ},$$

whence $$ji\,\overline{ON} = j\,\overline{OZ} = \overline{OE} = -\overline{OW} = -k\,\overline{ON}.$$

71. From this it appears that

and similarly

$$\left.\begin{aligned} ji &= -k, \\ kj &= -i, \\ ik &= -j, \end{aligned}\right\} \quad \text{........................... (12)}$$

and thus, by comparing (11),

$$\left.\begin{aligned} ij &= -ji = k, \\ jk &= -kj = i, \\ ki &= -ik = j. \end{aligned}\right\} \;((11), (12)).$$

These equations, along with

$$i^2 = j^2 = k^2 = -1 \quad ((7), (8), (9));$$

contain essentially the whole of Quaternions. But it is easy to see that, for the first group, we may substitute the single equation

$$ijk = -1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

since from it, by the help of the values of the squares of i, j, k, all the other expressions may be deduced. We may consider it proved in this way, or deduce it afresh from the figure above, thus

$$\begin{aligned} k\overline{ON} &= \overline{OW}, \\ jk\overline{ON} &= j\overline{OW} = \overline{OZ}, \\ ijk\overline{ON} &= ij\overline{OW} = i\overline{OZ} = \overline{OS} = -\overline{ON}. \end{aligned}$$

72. One most important step remains to be made, to wit the assumption referred to in § 64. We have treated i, j, k simply as quadrantal versors; and I, J, K as unit-vectors at right angles to each other, and coinciding with the axes of rotation of these versors. But if we collate and compare the equations just proved we have

$$\left\{\begin{aligned} ij &= k, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11) \\ iJ &= K, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}\right.$$

$$\left\{\begin{aligned} ji &= -k, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12) \\ jI &= -K, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10) \end{aligned}\right.$$

with the other similar groups symmetrically derived from them. Now the meanings we have assigned to i, j, k are quite independent of, and not inconsistent with, those assigned to I, J, K. And it is superfluous to use two sets of characters when one will suffice. Hence it appears that i, j, k may be substituted for I, J, K; in other words, *a unit-vector when employed as a factor may be considered as a quadrantal versor whose plane is perpendicular to the vector.* This is one of the main elements of the singular simplicity of the quaternion calculus.

73. Thus *the product, and therefore the quotient, of two perpendicular vectors is a third vector perpendicular to both.*

Hence the reciprocal (§ 51) of a vector is a vector which has the *opposite* direction to that of the vector, and its length is the reciprocal of the length of the vector.

The conjugate (§ 52) of a vector is simply the vector reversed. Hence, by § 52, if α be a vector

$$(T\alpha)^2 = \alpha K\alpha = \alpha(-\alpha) = -\alpha^2.$$

74. We may now see that *every versor may be represented by a power of a unit-vector.*

For, if α be any vector perpendicular to i (which is *any* definite unit-vector),

$i\alpha$, $= \beta$, is a vector equal in length to α, but perpendicular to both i and α;

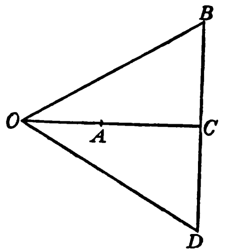
$$i^2\alpha = - \alpha,$$
$$i^3\alpha = -i\alpha = -\beta,$$
$$i^4\alpha = -i\beta = -i^2\alpha = \alpha.$$

Thus, by successive applications of i, α is turned round i as an axis through successive right angles. Hence it is natural to *define i^m as a versor which turns any vector perpendicular to i through m right angles in the positive direction of rotation about i as an axis.* Here m may have any rèal value whatever, for it is easily seen that analogy leads us to interpret a negative value of m as corresponding to rotation in the negative direction.

75. From this again it follows that *any quaternion may be exvressed as a power of a vector.* For the tensor and versor elements of the vector may be so chosen that, when raised to the same power, the one may be the tensor and the other the versor of the given quaternion. The vector must be, of course, perpendicular to the plane of the quaternion.

76. And we now see, as an immediate result of the last two sections, that the index-law holds with regard to powers of a quaternion (§ 63).

77. So far as we have yet considered it, a quaternion has been regarded as the *product* of a tensor and a versor: we are now to consider it as a *sum*. The easiest method of so analysing it seems to be the following.



Let $\frac{\overline{OB}}{\overline{OA}}$ represent any quaternion. Draw BC perpendicular to OA, produced if necessary.

Then, § 19, $\overline{OB} = \overline{OC} + \overline{CB}$.

But, § 22, $\overline{OC} = x\overline{OA}$

where x is a number, whose sign is the same as that of the cosine of $\angle AOB$.

Also, § 73, since CB is perpendicular to OA,

$$\overline{CB} = \gamma\overline{OA},$$

where γ is a vector perpendicular to OA and CB, i. e. to the plane of the quaternion.

Hence $$\frac{\overline{OB}}{\overline{OA}} = \frac{x\overline{OA} + \gamma\overline{OA}}{\overline{OA}} = x + \gamma.$$

Thus a quaternion, in general, may be decomposed into the sum of two parts, one numerical, the other a vector. Hamilton calls them the SCALAR, and the VECTOR, and denotes them respectively by the letters S and V prefixed to the expression for the quaternion.

78. Hence $q = Sq + Vq$, and if in the above example

$$\frac{\overline{OB}}{\overline{OA}} = q,$$

then $$\overline{OB} = \overline{OC} + \overline{CB} = Sq \, . \, \overline{OA} + Vq \, . \, \overline{OA}.$$

[The points are inserted to shew that S and V apply only to q, and not to $q\overline{OA}$.]

The equation above gives

$$\overline{OC} = Sq.\overline{OA},$$
$$\overline{CB} = Vq.\overline{OA}.$$

79. If, in the figure of last section, we produce BC to D, so as to double its length, and join OD, we have by § 52,

$$\frac{\overline{OD}}{\overline{OA}} = Kq = SKq + VKq;$$

$$\therefore \quad \overline{OD} = \overline{OC} + \overline{CD} = SKq.\overline{OA} + VKq.\overline{OA}.$$

Hence $\overline{OC} = SKq.\overline{OA},$

and $\overline{CD} = VKq.\overline{OA}.$

Comparing this value of $\overline{OC}$ with that in last section, we find

$$SKq = Sq, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

or *the scalar of the conjugate of a quaternion is equal to the scalar of the quaternion.*

Again, $\overline{CD} = -\overline{CB}$ by the figure, and the substitution of their values gives

$$VKq = -Vq, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

or *the vector of the conjugate of a quaternion is the vector of the quaternion reversed.*

[We may remark that the results of this section are simple consequences of the fact that the symbols S, V, K are commutative*. Thus

$$SKq = KSq = Sq,$$

since the conjugate of a number is the number itself; and

$$VKq = KVq = -Vq \ (\S\ 73).$$

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* It is curious to compare the properties of these quaternion symbols with those of the Elective Symbols of Logic, as given in BOOLE's wonderful treatise on the *Laws of Thought;* and to think that the same grand science of mathematical analysis, by processes remarkably similar to each other, reveals to us truths in the science of *position* far beyond the powers of the geometer, and truths of deductive reasoning to which unaided thought could never have led the logician.

Again, it is obvious that

$$\Sigma Sq = S\Sigma q, \quad \Sigma Vq = V\Sigma q,$$

and thence $$\Sigma Kq = K\Sigma q.]$$

80. Since any vector whatever may be represented by

$$xi + yj + zk$$

where x, y, z are numbers (or Scalars), and i, j, k may be any three non-coplanar vectors, §§ 23, 25—though they are usually understood as representing a rectangular system of unit-vectors—and since any scalar may be denoted by w; we may write, for any quaternion q, the expression

$$q = w + xi + yj + zk \quad (\S\ 78).$$

Here we have the essential dependence on four distinct numbers, from which the quaternion derives its name, exhibited in the most simple form.

And now we see at once that an equation such as

$$q' = q,$$

where $$q' = w' + x'i + y'j + z'k,$$

involves, of course, the *four* equations

$$w' = w, \quad x' = x, \quad y' = y, \quad z' = z.$$

81. We proceed to indicate another mode of proof of the distributive law of multiplication.

We have already defined, or assumed (§ 61), that

$$\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} = \frac{\beta + \gamma}{\alpha},$$

or $$\beta\alpha^{-1} + \gamma\alpha^{-1} = (\beta + \gamma)\alpha^{-1},$$

and have thus been able to understand what is meant by adding two quaternions.

But, writing α for α^{-1}, we see that this involves the equality

$$(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha;$$

from which, by taking the conjugates of both sides, we derive

$$\alpha'(\beta' + \gamma') = \alpha'\beta' + \alpha'\gamma' \quad (\S\ 55).$$

And a combination of these results (putting $\beta+\gamma$ for α' in the latter, for instance) gives

$$(\beta+\gamma)(\beta'+\gamma') = (\beta+\gamma)\beta'+(\beta+\gamma)\gamma'$$
$$= \beta\beta'+\gamma\beta'+\beta\gamma'+\gamma\gamma' \text{ by the former.}$$

Hence *the distributive principle is true in the multiplication of vectors.*

It only remains to show that it is true as to the scalar and vector parts of a quaternion, and then we shall easily attain the general proof.

Now, if a be any scalar, α any vector, and q any quaternion,

$$(a+\alpha)q = aq+\alpha q.$$

For, if β be the vector in which the plane of q is intersected by a plane perpendicular to α, we can find other two vectors, γ and δ, in these planes such that

$$\alpha = \frac{\gamma}{\beta}, \qquad q = \frac{\beta}{\delta}.$$

And, of course, a may be written $\frac{a\beta}{\beta}$; so that

$$(a+\alpha)q = \frac{a\beta+\gamma}{\beta}\cdot\frac{\beta}{\delta} = \frac{a\beta+\gamma}{\delta}$$
$$= a\frac{\beta}{\delta}+\frac{\gamma}{\delta} = a\frac{\beta}{\delta}+\frac{\gamma}{\beta}\cdot\frac{\beta}{\delta}$$
$$= aq+\alpha q.$$

And the conjugate may be written

$$q'(a'+\alpha') = q'a'+q'\alpha' \quad (\S\ 55).$$

Hence, generally,

$$(a+\alpha)(b+\beta) = ab+a\beta+b\alpha+\alpha\beta\,;$$

or, breaking up a and b each into the sum of two scalars, and α, β each into the sum of two vectors,

$$(a_1+a_2+\alpha_1+\alpha_2)(b_1+b_2+\beta_1+\beta_2)$$
$$= (a_1+a_2)(b_1+b_2)+(a_1+a_2)(\beta_1+\beta_2)+(b_1+b_2)(\alpha_1+\alpha_2)$$
$$+(\alpha_1+\alpha_2)(\beta_1+\beta_2$$

(by what precedes, all the factors on the right are distributive, so that we may easily put it in the form)

$$= (a_1+\alpha_1)(b_1+\beta_1)+(a_1+\alpha_1)(b_2+\beta_2)+(a_2+\alpha_2)(b_1+\beta_1)$$
$$+(a_2+\alpha_2)(b_2+\beta_2).$$

Putting $a_1+\alpha_1=p,\ a_2+\alpha_2=q,\ b_1+\beta_1=r,\ b_2+\beta_2=s,$

we have

$$(p+q)(r+s) = pr+ps+qr+qs.$$

82. For variety, we shall now for a time forsake the geometrical mode of proof we have hitherto adopted, and deduce some of our next steps from the analytical expression for a quaternion given in § 80, and the properties of a rectangular system of unit-vectors as in § 71.

We will commence by proving the result of § 77 anew.

83. Let

$$\alpha = xi+yj+zk,$$
$$\beta = x'i+y'j+z'k.$$

Then, because by § 71 every product or quotient of i, j, k is reducible to one of them or to a number, we are entitled to assume

$$q = \frac{\beta}{\alpha} = \omega+\xi i+\eta j+\zeta k,$$

where ω, ξ, η, ζ are numbers. This is the proposition of § 80.

84. But it may be interesting to find ω, ξ, η, ζ in terms of x, y, z, x', y', z'.

We have

$$\beta = q\alpha,$$

or

$$x'i+y'j+z'k = (\omega+\xi i+\eta j+\zeta k)(xi+yj+zk)$$
$$= -(\xi x+\eta y+\zeta z)+(\omega x+\eta z-\zeta y)i+(\omega y+\zeta x-\xi z)j+(\omega z+\xi y-\eta x)k,$$

as we easily see by the expressions for the powers and products of i, j, k, given in § 71. But the student must pay particular attention to the *order* of the factors, else he is certain to make mistakes.

This (§ 80) resolves itself into the four equations

$$\begin{aligned} 0 &= \qquad \xi x + \eta y + \zeta z, \\ x' &= \omega x \qquad + \eta z - \zeta y, \\ y' &= \omega y - \xi z \qquad + \zeta x, \\ z' &= \omega z + \xi y - \eta x. \end{aligned}$$

The three last equations give

$$xx' + yy' + zz' = \omega(x^2 + y^2 + z^2),$$

which determines ω.

Also we have, from the same three,

$$\xi x' + \eta y' + \zeta z' = 0;$$

which, combined with the first, gives

$$\frac{\xi}{yz' - zy'} = \frac{\eta}{zx' - xz'} = \frac{\zeta}{xy' - yx'};$$

and the common value of these three fractions is then easily seen to be

$$\frac{1}{x^2 + y^2 + z^2}.$$

It is easy enough to interpret these expressions by means of ordinary cöordinate geometry: but a much simpler process will be furnished by quaternions themselves in the next chapter, and, in giving it, we shall refer back to this section.

85. The associative law of multiplication is now to be proved by means of the distributive (§ 81). We leave the proof to the student. He has merely to multiply together the factors

$$w + xi + yj + zk, \quad w' + x'i + y'j + z'k, \quad \text{and} \quad w'' + x''i + y''j + z''k,$$

as follows:—

First, multiply the third factor by the second, and then multiply the product by the first; next, multiply the second factor by the first and employ the product to multiply the third:

always remembering that the multiplier in any product is placed *before* the multiplicand. He will find the scalar parts and the coefficients of i, j, k, in these products, respectively equal, each to each.

86. With the same expressions for α, β, as in section 83, we have

$$\begin{aligned}\alpha\beta &= (xi+yj+zk)(x'i+y'j+z'k)\\ &= -(xx'+yy'+zz')+(yz'-zy')i+(zx'-xz')j+(xy'-yx')k.\end{aligned}$$

But we have also

$$\beta\alpha = -(xx'+yy'+zz')-(yz'-zy')i-(zx'-xz')j-(xy'-yx')k.$$

The only difference is in the *sign* of the vector parts.

Hence
$$S\alpha\beta = S\beta\alpha, \quad\text{.........}\quad (1)$$
$$V\alpha\beta = -V\beta\alpha, \quad\text{.........}\quad (2)$$

also
$$\alpha\beta+\beta\alpha = 2\,S\alpha\beta, \quad\text{.........}\quad (3)$$
$$\alpha\beta-\beta\alpha = 2\,V\alpha\beta, \quad\text{.........}\quad (4)$$

and, finally, by § 79,
$$\alpha\beta = K\beta\alpha. \quad\text{.........}\quad (5)$$

87. If $\alpha = \beta$ we have of course (§ 25)

$$x = x', \qquad y = y', \qquad z = z',$$

and the formulae of last section become

$$\alpha\beta = \beta\alpha = \alpha^2 = -(x^2+y^2+z^2);$$

which was anticipated in § 73, where we proved the formula

$$(T\alpha)^2 = -\alpha^2,$$

and also, to a certain extent, in § 25.

88. Now let q and r be any quaternions, then

$$\begin{aligned}Sqr &= S.(Sq+Vq)(Sr+Vr),\\ &= S.(SqSr+Sr.Vq+Sq.Vr+Vq\,Vr),\\ &= SqSr+S.Vq\,Vr,\end{aligned}$$

since the two middle terms are vectors.

Similarly, $Srq = Sr\,Sq + S.Vr\,Vq.$

Hence, since by (1) of § 86 we have

$$S.VqVr = S.VrVq,$$

we see that

$$Sqr = Srq, \quad\text{..............................}\quad (1)$$

a formula of considerable importance.

It may easily be extended to any number of quaternions, because, r being arbitrary, we may put for it rs. Thus we have (putting a dot after the S to shew that it refers to the whole product that follows it)

$$\begin{aligned} S.qrs &= S.rsq \\ &= S.sqr \end{aligned}$$

by a second application of the process. In words, we have the theorem—*the scalar of the product of any number of given quaternions depends only upon the cyclical order in which they are arranged.*

89. An important case is that of three factors, each a vector. The formula then becomes

$$S.\alpha\beta\gamma = S.\beta\gamma\alpha = S.\gamma\alpha\beta.$$

But

$$\begin{aligned} S.\alpha\beta\gamma &= S.\alpha(S\beta\gamma + V\beta\gamma) \\ &= S.\alpha V\beta\gamma, \qquad \text{since } \alpha S\beta\gamma \text{ is a vector,} \\ &= -S.\alpha V\gamma\beta, \qquad \text{by (2) of § 86,} \\ &= -S.\alpha(S\gamma\beta + V\gamma\beta) \\ &= -S.\alpha\gamma\beta. \end{aligned}$$

Hence *the scalar of the product of three vectors changes sign when the cyclical order is altered.*

Other curious propositions connected with this will be given later, as we wish to devote this chapter to the production of the fundamental formulae in as compact a form as possible.

90. By (4) of § 86,

$$2V\beta\gamma = \beta\gamma - \gamma\beta.$$

Hence

$$2V.\alpha V\beta\gamma = V.\alpha(\beta\gamma - \gamma\beta)$$

(by multiplying both by α, and taking the vector parts of each side)

$$= V(\alpha\beta\gamma + \beta\alpha\gamma - \beta\alpha\gamma - \alpha\gamma\beta)$$

(by introducing the null term $\beta\alpha\gamma - \beta\alpha\gamma$).

That is

$$\begin{aligned} 2\,V.\alpha V\beta\gamma &= V.(\alpha\beta + \beta\alpha)\gamma - V.(\beta S\alpha\gamma + \beta V\alpha\gamma + S\alpha\gamma.\beta + V\alpha\gamma.\beta) \\ &= V.(2\,S\alpha\beta)\gamma - 2\,V.\beta S\alpha\gamma \end{aligned}$$

(if we notice that $V.V\alpha\gamma.\beta = -V.\beta V\alpha\gamma$, by (2) of § 86).

Hence $$V.\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\gamma\alpha, \quad\text{......................}\quad (1)$$

a formula of constant occurrence.

Adding $\alpha S\beta\gamma$ to both sides we get another most valuable formula

$$V.\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta\,; \quad\text{..................}\quad (2)$$

and the form of this shews that we may interchange γ and α without altering the right-hand member. This gives

$$V.\alpha\beta\gamma = V.\gamma\beta\alpha,$$

a formula which may be greatly extended.

91. We have also

$$\begin{aligned} V.V\alpha\beta V\gamma\delta &= -V.V\gamma\delta V\alpha\beta \qquad \text{by (2) of § 86:} \\ &= \delta S.\gamma V\alpha\beta - \gamma S.\delta V\alpha\beta = \delta\,S.\alpha\beta\gamma - \gamma S.\alpha\beta\delta, \\ &= -\beta S.\alpha V\gamma\delta + \alpha S.\beta V\gamma\delta = -\beta S.\alpha\gamma\delta + \alpha S.\beta\gamma\delta, \end{aligned}$$

all of these being arrived at by the help of § 90 (1) and of § 89; and by treating alternately $V\alpha\beta$ and $V\gamma\delta$ as *simple* vectors.

Equating two of these values, we have

$$\delta S.\alpha\beta\gamma = \alpha S.\beta\gamma\delta + \beta S.\gamma\alpha\delta + \gamma S.\alpha\beta\delta, \quad\text{...........}\quad (3)$$

a very useful formula, expressing any vector whatever in terms of three given vectors.

92. That such an expression is possible we knew already by § 23. For variety we may seek another expression of a similar character, by a process which differs entirely from that employed in last section.

α, β, γ being any three vectors, we may derive from them three others $V\alpha\beta$, $V\beta\gamma$, $V\gamma\alpha$; and, as these will not generally be coplanar, any other vector δ may be expressed as the sum of the three, each multiplied by some scalar (§ 23). It is required to find this expression for δ.

Let $$\delta = xV\alpha\beta + yV\beta\gamma + zV\gamma\alpha.$$

Then $$S\gamma\delta = xS.\gamma\alpha\beta = xS\alpha\beta\gamma,$$

the terms in y and z going out, because

$$S.\gamma V\beta\gamma = S.\gamma\beta\gamma = S\beta\gamma^2 = \gamma^2 S\beta = 0,$$

for γ^2 is (§ 73) a number.

Similarly $$S\beta\delta = zS.\beta\gamma\alpha = zS.\alpha\beta\gamma,$$

and $$S\alpha\delta = yS.\alpha\beta\gamma.$$

Thus $$\delta S.\alpha\beta\gamma = V\alpha\beta S\gamma\delta + V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta. \quad \text{.........} (4)$$

93. We conclude the chapter by showing (as promised in § 64) that the assumption that the product of two parallel vectors is a number, and that of two perpendicular vectors a third vector perpendicular to both, is not only useful and convenient but absolutely inevitable if our system is to deal indifferently with all directions in space. We abridge Hamilton's reasoning.

Suppose that there is no direction in space pre-eminent, and that the product of two vectors is something which has quantity, so as to vary in amount if the factors are changed, and to have its sign changed if that of one of them is reversed; if the vectors be parallel, their product cannot be, in whole or in part, a vector *inclined* to them, for there is nothing to determine the direction in which it must lie. It cannot be a vector *parallel* to them; for by changing the sign of both factors the product is unchanged, whereas, as the whole system has been reversed, the product vector ought to have been reversed. Hence it must be a number. Again, the product of two perpendicular vectors cannot be wholly or partly a number, because on inverting one of them the sign of that number ought to change; but inverting one of them is simply equivalent to a rotation through two right angles about the other, and (from the symmetry of space) ought to leave the number unchanged. Hence the product of two perpendicular vectors must be a vector, and an easy extension of the same reasoning shows that it must be perpendicular to each of the factors. It is easy to carry this farther, but enough has been said to show the character of the reasoning.

EXAMPLES TO CHAPTER II.

1. It is obvious from the properties of polar triangles that any mode of representing versors by the *sides* of a triangle must have an equivalent statement in which they are represented by *angles* in the polar triangle.

Show directly that the product of two versors represented by two angles of a spherical triangle is a third versor represented by the *supplement* of the remaining angle of the triangle; and determine the rule which connects the *directions* in which these angles are to be measured.

2. Hence derive another proof that we have not generally

$$pq = qp.$$

3. Hence show that the proof of the associative principle, § 57, may be made to depend upon the fact that if from any point of the sphere tangent arcs be drawn to a spherical conic,

and also arcs to the foci, the inclination of either tangent arc to one of the focal arcs is equal to that of the other tangent arc to the other focal arc.

4. Prove the formulae

$$2\,S.\alpha\beta\gamma = \alpha\beta\gamma - \gamma\beta\alpha,$$
$$2\,V.\alpha\beta\gamma = \alpha\beta\gamma + \gamma\beta\alpha.$$

5. Show that, whatever odd number of vectors be represented by α, β, γ, &c., we have always

$$V.\alpha\beta\gamma\delta\epsilon = V.\epsilon\delta\gamma\beta\alpha,$$
$$V.\alpha\beta\gamma\delta\epsilon\zeta\eta = V.\eta\zeta\epsilon\delta\gamma\beta\alpha, \text{ \&c.}$$

6. Show that

$$S.V\alpha\beta V\beta\gamma V\gamma\alpha = -(S.\alpha\beta\gamma)^2,$$
$$V.V\alpha\beta V\beta\gamma V\gamma\alpha = V\alpha\beta(\gamma^2 S\alpha\beta - S\beta\gamma S\gamma\alpha) + \;\ldots\ldots,$$

and $$V.(V\alpha\beta V.V\beta\gamma V\gamma\alpha) = (\beta S\alpha\gamma - \alpha S\beta\gamma)S.\alpha\beta\gamma.$$

7. If α, β, γ be any vectors at right angles to each other, show that

$$(\alpha^2 + \beta^2 + \gamma^2)S.\alpha\beta\gamma = \alpha^4 V\beta\gamma + \beta^4 V\gamma\alpha + \gamma^4 V\alpha\beta.$$

8. If α, β, γ be non-coplanar vectors, find the relations among the six scalars, x, y, z and ξ, η, ζ, which are implied in the equation

$$x\alpha + y\beta + z\gamma = \xi V\beta\gamma + \eta V\gamma\alpha + \zeta V\alpha\beta.$$

9. If α, β, γ be any three non-coplanar vectors, express any fourth vector, δ, as a linear function of each of the following sets of three derived vectors,

$$V.\gamma\alpha\beta, \quad V.\alpha\beta\gamma, \quad V.\beta\gamma\alpha,$$

and $$V.V\alpha\beta V\beta\gamma V\gamma\alpha, \quad V.V\beta\gamma V\gamma\alpha V\alpha\beta, \quad V.V\gamma\alpha V\alpha\beta V\beta\gamma.$$

10. Eliminate ρ from the equations

$$S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c, \quad S\delta\rho = d,$$

where α, β, γ, δ are vectors, and a, b, c, d scalars.

CHAPTER III.

INTERPRETATIONS AND TRANSFORMATIONS OF QUATERNION EXPRESSIONS.

94. AMONG the most useful characteristics of the Calculus of Quaternions the ease of interpreting its formulæ geometrically, and the extraordinary variety of transformations of which the simplest expressions are susceptible, deserve a prominent place. We devote this Chapter to the more simple of these, together with a few of somewhat more complex character but of constant occurrence in geometrical and physical investigations. Others will appear in every succeeding Chapter. It is here, perhaps, that the student is likely to feel most strongly the peculiar difficulties of the new Calculus. But on that very account he should endeavour to master them, for the variety of forms which any one formula may assume, though puzzling to the beginner, is of the most extraordinary advantage to the advanced student, not alone as aiding him in the solution of complex questions, but as affording an invaluable mental discipline.

95. If we refer again to the figure of § 77 we see that

$$OC = OB \cos AOB,$$

$$CB = OB \sin AOB.$$

Hence, if $\overline{OA} = \alpha$, $\overline{OB} = \beta$, and $\angle AOB = \theta$, we have

$$OB = T\beta, \qquad OA = T\alpha,$$

$$OC = T\beta \cos\theta, \quad CB = T\beta \sin\theta.$$

Hence $$S\frac{\beta}{\alpha} = \frac{OC}{OA} = \frac{T\beta}{T\alpha}\cos\theta.$$

Similarly $$TV\frac{\beta}{\alpha} = \frac{CB}{OA} = \frac{T\beta}{T\alpha}\sin\theta.$$

Hence, if ϵ be a unit-vector perpendicular to α and β, or

$$\epsilon = \frac{U\overline{CB}}{U\overline{OA}} = UV\frac{\beta}{\alpha},$$

we have $$V\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}\sin\theta\,\epsilon.$$

96. In the same way we may shew that

$$\begin{aligned} S\alpha\beta &= -T\alpha\, T\beta\cos\theta, \\ TV\alpha\beta &= \quad T\alpha\, T\beta\sin\theta, \\ \text{and} \qquad V\alpha\beta &= \quad T\alpha\, T\beta\sin\theta\,\eta \\ \text{where} \qquad \eta &= \quad UV\alpha\beta = -UV\frac{\beta}{\alpha}. \end{aligned}$$

Thus *the scalar of the product of two vectors is the continued product of their tensors and of the cosine of the supplement of the contained angle.*

The tensor of the vector of the product of two vectors is the continued product of their tensors and the sine of the contained angle; and the versor of the same is a unit-vector perpendicular to both, and such that the rotation about it from the first vector to the second is right-handed or negative.

Hence $TV\alpha\beta$ is double the area of the triangle two of whose sides are α, β.

97.

(*a*.) In any triangle ABC we have

$$\overline{AC} = \overline{AB} + \overline{BC}.$$

Hence $$\overline{AC}^2 = S\,\overline{AC}\,\overline{AC} = S.\overline{AC}(\overline{AB} + \overline{BC}).$$

With the usual notation for a plane triangle the interpretation of this formula is

$$-b^2 = -bc\cos A - ab\cos C,$$

or $$b = a\cos C + c\cos A.$$

(*b.*) Again we have, obviously,

$$V.\overline{AB}\,\overline{AC} = V.\overline{AB}(\overline{AB}+\overline{BC})$$
$$= V.\overline{AB}\,\overline{BC},$$

or $$cb\sin A = ca\sin B,$$

whence $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

These are truths, but not truisms, as we might have been led to fancy from the excessive simplicity of the process employed.

98. From § 96 it follows that, if α and β be both actual (i. e. non-evanescent) vectors, the equation

$$S.\alpha\beta = 0$$

shews that $\cos\theta = 0$, or that α *is perpendicular to* β. And, in fact, we know already that the product of two perpendicular vectors is a vector.

Again, if $$V\alpha\beta = 0,$$

we must have $\sin\theta = 0$, or α *is parallel to* β. We know already that the product of two parallel vectors is a scalar.

Hence we see that $$S\alpha\beta = 0$$

is equivalent to $$\alpha = V\gamma\beta,$$

where γ is an undetermined vector; and that

$$V\alpha\beta = 0$$

is equivalent to $$\alpha = x\beta,$$

where x is an undetermined scalar.

99. If we write, as in § 83,

$$\alpha = ix + jy + kz,$$
$$\beta = ix' + jy' + kz',$$

we have, at once, by § 86,

$$S\alpha\beta = -xx' - yy' - zz'$$

$$= -rr'\left(\frac{x}{r}\frac{x'}{r'} + \frac{y}{r}\frac{y'}{r'} + \frac{z}{r}\frac{z'}{r'}\right)$$

where $r = \sqrt{x^2+y^2+z^2}$, $r' = \sqrt{x'^2+y'^2+z'^2}$.

Also $$V\alpha\beta = rr'\left\{\frac{yz'-zy'}{rr'}i + \frac{zx'-xz'}{rr'}j + \frac{xy'-yx'}{rr'}k\right\}.$$

These express in Cartesian coördinates the propositions we have just proved. In commencing the subject it may perhaps assist the student to see these more familiar forms for the quaternion expressions; and he will doubtless be induced by their appearance to prosecute the subject, since he cannot fail even at this stage to see how much more simple the quaternion expressions are than those to which he has been accustomed.

100. The expression $$S.\alpha\beta\gamma$$
may be written $$S.(V\alpha\beta)\gamma$$
because the quaternion $\alpha\beta\gamma$ may be broken up into
$$(S\alpha\beta)\gamma + (V\alpha\beta)\gamma$$
of which the first term is a vector.

But, by § 96,
$$S.(V\alpha\beta)\gamma = T\alpha\, T\beta \sin\theta\, S\eta\gamma.$$
Here $T\eta = 1$, let ϕ be the angle between η and γ, then finally
$$S.\alpha\beta\gamma = -T\alpha\, T\beta\, T\gamma \sin\theta\cos\phi.$$
But as η is perpendicular to α and β, $T\gamma\cos\phi$ is the length of the perpendicular from the extremity of γ upon the plane of α, β. And as the product of the other three factors is (§ 96) the area of the parallelogram two of whose sides are α, β, we see that the magnitude of $S.\alpha\beta\gamma$, independent of its sign, is *the volume of the parallelepiped of which three coördinate edges are* α, β, γ; or six times the volume of the pyramid which has α, β, γ for edges.

101. Hence the equation

$$S.\alpha\beta\gamma = 0,$$

if we suppose α, β, γ to be actual vectors, shews either that

$$\sin\theta = 0,$$

or

$$\cos\phi = 0,$$

i. e. *two of the three vectors are parallel*, or *all three lie in one plane.*

This is consistent with previous results, for if $\gamma = p\beta$ we have

$$S.\alpha\beta\gamma = p\,S.\alpha\beta^2 = 0\,;$$

and, if γ be coplanar with α, β, we have $\gamma = p\alpha+q\beta$, and

$$S.\alpha\beta\gamma = S.\alpha\beta(p\alpha+q\beta) = 0.$$

102. This property of the expression $S.\alpha\beta\gamma$ prepares us to find that it is a determinant. And, in fact, if we take α, β as in § 83, and in addition

$$\gamma = ix''+jy''+kz'',$$

we have at once

$$S.\alpha\beta\gamma = -x''(yz'-zy')-y''(zx'-xz')-z''(xy'-yx'),$$

$$= -\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}.$$

The determinant changes sign if we make any two rows change places. This is the proposition we met with before (§ 89) in the form

$$S.\alpha\beta\gamma = -S.\beta\alpha\gamma = S.\beta\gamma\alpha, \text{ \&c.}$$

If we take three new vectors

$$\alpha_1 = ix+jx'+kx'',$$
$$\beta_1 = iy+jy'+ky'',$$
$$\gamma_1 = iz+jz'+kz'',$$

we thus see that they are coplanar if α, β, γ are so. That is, if

$$S.\alpha\beta\gamma = 0,$$

then

$$S.\alpha_1\beta_1\gamma_1 = 0.$$

103. We have, by § 52,

$$(Tq)^2 = qKq = (Sq+Vq)(Sq-Vq) \quad (\S\ 79),$$
$$= (Sq)^2-(Vq)^2 \text{ by algebra,}$$
$$= (Sq)^2+(TVq)^2 \quad (\S\ 73).$$

If $q = \alpha\beta$, we have $Kq = \beta\alpha$, and the formula becomes

$$\alpha\beta.\beta\alpha = \alpha^2\beta^2 = (S\alpha\beta)^2-(V\alpha\beta)^2.$$

In Cartesian cöordinates this is

$$(x^2+y^2+z^2)(x'^2+y'^2+z'^2)$$
$$=(xx'+yy'+zz')^2+(yz'-zy')^2+(zx'-xz')^2+(xy'-yx')^2.$$

More generally we have

$$(T(qr))^2 = qr\,K(qr)$$
$$= qr\,Kr\,Kq \ (\S\ 55) = (Tq)^2(Tr)^2 \quad (\S\ 52).$$

If we write

$$q = w+\alpha = w+ix+jy+kz,$$
$$r = w'+\beta = w'+ix'+jy'+kz\,;$$

this becomes

$$(w^2+x^2+y^2+z^2)(w'^2+x'^2+y'^2+z'^2)$$
$$= (ww'-xx'-yy'-zz')^2+(wx'+w'x+yz'-zy')^2$$
$$+(wy'+w'y+zx'-xz')^2+(wz'+w'z+xy'-yx')^2,$$

a formula of algebra due to Euler.

104. We have, of course, by multiplication,

$$(\alpha+\beta)^2 = \alpha^2+\alpha\beta+\beta\alpha+\beta^2 = \alpha^2+2S\alpha\beta+\beta^2 \quad (\S\ 86\ (3)).$$

Translating into the usual notation of plane trigonometry, this becomes

$$c^2 = a^2-2\,ab\cos C+b^2,$$

the common formula.

Again, $V(\alpha+\beta)(\alpha-\beta) = -V\alpha\beta+V\beta\alpha = -2\,V\alpha\beta$ (§ 86 (2)).

Taking tensors of both sides we have the theorem, *the parallelogram whose sides are parallel and equal to the diagonals of a given parallelogram, has double its area* (§ 96).

Also $$S(\alpha+\beta)(\alpha-\beta) = \alpha^2-\beta^2,$$

and vanishes only when $\alpha^2 = \beta^2$, or $T\alpha = T\beta$; that is, *the diagonals of a parallelogram are at right angles to one another, when, and only when, it is a rhombus.*

Later it will be shewn that this contains a proof that the angle in a semicircle is a right angle.

105. The expression
$$\rho = \alpha\beta\alpha^{-1}$$
obviously denotes a vector whose tensor is equal to that of β.

But we have
$$S.\beta\alpha\rho = 0,$$
so that ρ is in the plane of α, β.

Also we have
$$S\alpha\rho = S\alpha\beta,$$
so that β and ρ make equal angles with α, evidently on opposite sides of it. Thus if α be the perpendicular to a reflecting surface and β the path of an incident ray, ρ will be the path of the reflected ray.

Another mode of obtaining these results is to expand the above expression, thus, § 90 (2)
$$\begin{aligned}\rho &= 2\alpha^{-1}S\alpha\beta-\beta\\ &= 2\alpha^{-1}S\alpha\beta-\alpha^{-1}(S\alpha\beta+V\alpha\beta)\\ &= \alpha^{-1}(S\alpha\beta-V\alpha\beta),\end{aligned}$$
so that in the figure of § 77 we see that if $\overline{OA}=\alpha$, and $\overline{OB}=\beta$, we have $\overline{OD}=\rho=\alpha\beta\alpha^{-1}$.

106. For any three coplanar vectors the expression
$$\rho = \alpha\beta\gamma$$
is (§ 101) a vector. It is interesting to determine what this vector is. The reader will easily see that if a circle be described about the triangle, two of whose sides are (in order) α and β, and if from the extremity of β a line parallel to γ be drawn

again cutting the circle, the vector joining the point of intersection with the origin of α is the direction of the vector $\alpha\beta\gamma$. For we may write it in the form

$$\rho = \alpha\beta^2\beta^{-1}\gamma = -(T\beta)^2\,\alpha\beta^{-1}\gamma = -(T\beta)^2\,\frac{\alpha}{\beta}\gamma,$$

which shows that the *versor* which turns β into a direction parallel to α, turns γ into a direction parallel to ρ. And this is the long known property of opposite angles of a quadrilateral inscribed in a circle.

Hence if α, β, γ be the sides of a triangle taken in order, the tangents to the circumscribing circle at the angles of the triangle are parallel respectively to

$$\alpha\beta\gamma, \quad \beta\gamma\alpha, \quad \text{and} \quad \gamma\alpha\beta.$$

Suppose two of these to be parallel, i. e. let

$$\alpha\beta\gamma = x\beta\gamma\alpha = x\alpha\gamma\beta \quad (\S\ 90),$$

since the expression is a vector. Hence

$$\beta\gamma = x\gamma\beta,$$

which requires either

$$x = 1, \quad V\gamma\beta = 0, \quad \text{or} \quad \gamma \parallel \beta,$$

a case not contemplated in the problem;

$$\text{or} \qquad x = -1, \qquad S\beta\gamma = 0,$$

i. e. the triangle is right-angled. And geometry shews us at once that this is correct.

Again, if the triangle be isosceles, the tangent at the vertex is parallel to the base. Here we have

$$x\beta = \alpha\beta\gamma,$$

$$\text{or} \qquad x(\alpha+\gamma) = \alpha(\alpha+\gamma)\gamma;$$

whence $x = \gamma^2 = \alpha^2$, or $T\gamma = T\alpha$, as required.

As an elegant extension of this proposition the reader may prove that the vector of the continued product $\alpha\beta\gamma\delta$ of the vector-sides of a quadrilateral inscribed in a sphere is parallel to the radius drawn to the corner (α, δ).

107. To exemplify the variety of possible transformations even of simple expressions, we will take two cases which are of frequent occurrence in applications to geometry.

Thus $$T(\rho+\alpha)=T(\rho-\alpha),$$

[which expresses that if

$$\overline{OA}=\alpha,\quad \overline{OA'}=-\alpha,\quad \text{and}\quad \overline{OP}=\rho,$$

we have $$AP=A'P,$$

and thus that P is any point equidistant from two fixed points,] may be written

$$(\rho+\alpha)^2=(\rho-\alpha)^2,$$

or $$\rho^2+2S\alpha\rho+\alpha^2=\rho^2-2S\alpha\rho+\alpha^2 \quad (\S\ 104),$$

whence $$S\alpha\rho=0.$$

This may be changed to

$$\alpha\rho+\quad\rho\alpha=0,$$

or $$\alpha\rho+K\alpha\rho=0,$$

$$SU\frac{\rho}{\alpha}=0,$$

or finally, $$TVU\frac{\rho}{\alpha}=1,$$

all of which express properties of a plane.

Again, $$T\rho=T\alpha$$

may be written $$T\frac{\rho}{\alpha}=1,$$

$$\left(S\frac{\rho}{\alpha}\right)^2-\left(V\frac{\rho}{\alpha}\right)^2=1,$$

$$(\rho+\alpha)^2-2S\alpha(\rho+\alpha)=0,$$

$$\rho=(\rho+\alpha)^{-1}\alpha(\rho+\alpha),$$

$$S(\rho+\alpha)(\rho-\alpha)=0, \text{ or finally,}$$

$$T.(\rho+\alpha)(\rho-\alpha)=2TV\alpha\rho.$$

All of these express properties of a sphere. They will be interpreted when we come to geometrical applications.

108. We have seen in § 95 that a quaternion may be divided into its scalar and vector parts as follows:—

$$\frac{\beta}{a} = S\frac{\beta}{a} + V\frac{\beta}{a} = \frac{T\beta}{Ta}(\cos\theta + \epsilon\sin\theta);$$

where θ is the angle between the directions of a and β, and $\epsilon = UV\frac{\beta}{a}$ is the unit-vector perpendicular to the plane of a and β so situated that positive (i. e. left-handed) rotation about it turns a towards β.

Similarly we have (§ 96)

$$\begin{aligned} a\beta &= Sa\beta + Va\beta \\ &= TaT\beta(-\cos\theta - \epsilon\sin\theta), \end{aligned}$$

θ and ϵ having the same signification as before.

109. Hence, considering the versor parts alone, we have

$$U\frac{\beta}{a} = -Ua\beta = \cos\theta + \epsilon\sin\theta.$$

Similarly $$U\frac{\gamma}{\beta} = \cos\phi + \epsilon\sin\phi;$$

ϕ being the positive angle between the directions of γ and β, and ϵ the same vector as before, if a, β, γ be coplanar.

Also we have

$$U\frac{\gamma}{a} = \cos(\theta+\phi) + \epsilon\sin(\theta+\phi).$$

But we have always

$$\frac{\gamma}{\beta}.\frac{\beta}{a} = \frac{\gamma}{a}, \text{ and therefore}$$

$$U\frac{\gamma}{\beta}.U\frac{\beta}{a} = U\frac{\gamma}{a};$$

or $$\begin{aligned} \cos(\phi+\theta) + \epsilon\sin(\phi+\theta) &= (\cos\phi + \epsilon\sin\phi)(\cos\theta + \epsilon\sin\theta) \\ &= \cos\phi\cos\theta - \sin\phi\sin\theta + \epsilon(\sin\phi\cos\theta + \cos\phi\sin\theta), \end{aligned}$$

from which we have at once the fundamental formulæ for the cosine and sine of the sum of two arcs, by equating separately the scalar and vector parts of these quaternions.

And we see, as an immediate consequence of the expressions above, that

$$\cos m\theta + \epsilon \sin m\theta = (\cos\theta + \epsilon \sin\theta)^m$$

if m be a positive whole number. For the left-hand side is a versor which turns through the angle $m\theta$ at once, while the right-hand side is a versor which effects the same object by m successive turnings each through an angle θ. See § 8.

110. To extend this proposition to fractional indices we have only to write $\frac{\theta}{n}$ for θ, when we obtain the results as in ordinary trigonometry.

From De Moivre's Theorem, thus proved, we may of course deduce the rest of Analytical Trigonometry. And as we have already deduced, as interpretations of self-evident quaternion transformations (§§ 97, 104), the fundamental formulæ for the solution of plane triangles, we will now pass to the consideration of spherical trigonometry, a subject specially adapted for treatment by quaternions; but to which we cannot afford more than a very few sections. The reader is referred to Hamilton's works for the treatment of this subject by quaternion exponentials.

111. Let α, β, γ be unit-vectors drawn from the centre to the corners A, B, C of a triangle on the unit-sphere. Then it is evident that, with the usual notation, we have (§ 96),

$$S\alpha\beta = -\cos c, \qquad S\beta\gamma = -\cos a, \qquad S\gamma\alpha = -\cos b,$$

$$TV\alpha\beta = \sin c, \qquad TV\beta\gamma = \sin a, \qquad TV\gamma\alpha = \sin b.$$

Also $UV\alpha\beta$, $UV\beta\gamma$, $UV\gamma\alpha$ are evidently the vectors of the corners of the polar triangle.

Hence $$S.UV\alpha\beta UV\beta\gamma = \cos B, \text{ \&c.},$$

$$TV.UV\alpha\beta UV\beta\gamma = \sin B, \text{ \&c.}$$

Now (§ 90 (1)) we have

$$SV\alpha\beta V\beta\gamma = S.\alpha V.\beta V\beta\gamma$$
$$= -S\alpha\beta S\beta\gamma + \beta^2 S\alpha\gamma.$$

Remembering that we have

$$SV\alpha\beta V\beta\gamma = TV\alpha\beta TV\beta\gamma S.UV\alpha\beta UV\beta\gamma,$$

we see that the formula just written is equivalent to

$$\sin a \sin c \cos B = -\cos a \cos c + \cos b,$$

or $\quad \cos b = \cos a \cos c + \sin a \sin c \cos B.$

112. Again,

$$V.V\alpha\beta V\beta\gamma = -\beta S\alpha\beta\gamma$$

which gives

$$TV.V\alpha\beta V\beta\gamma = S.\alpha\beta\gamma = S.\alpha V\beta\gamma = S.\beta V\gamma\alpha = S.\gamma V\alpha\beta,$$

or $\quad \sin a \sin c \sin B = \sin a \sin p_a = \sin b \sin p_b = \sin c \sin p_c$;

where p_a is the arc drawn from A perpendicular to BC, &c.

Hence
$$\sin p_a = \sin c \sin B,$$
$$\sin p_b = \frac{\sin a \sin c}{\sin b} \sin B,$$
$$\sin p_c = \sin a \sin B.$$

113. Combining the results of the last two sections, we have

$$V\alpha\beta.V\beta\gamma = \sin a \sin c \cos B - \beta \sin a \sin c \sin B$$
$$= \sin a \sin c (\cos B - \beta \sin B).$$

Hence
$$\left.\begin{aligned} U.V\alpha\beta V\beta\gamma &= (\cos B - \beta \sin B), \\ U.V\gamma\beta V\beta\alpha &= (\cos B + \beta \sin B). \end{aligned}\right\}$$
and

These are therefore versors which turn the system negatively or positively about OB through the angle B.

As another instance, we have

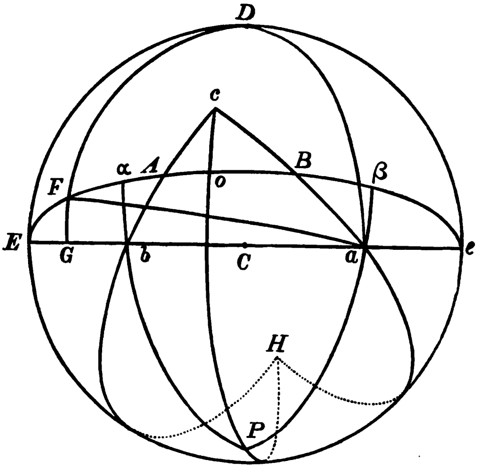
$$\tan B = \frac{\sin B}{\cos B} = \frac{TV.V\alpha\beta V\beta\gamma}{S.V\alpha\beta V\beta\gamma} = -\beta^{-1} \frac{V.V\alpha\beta V\beta\gamma}{S.V\alpha\beta V\beta\gamma}$$
$$= -\frac{S.\alpha\beta\gamma}{S\alpha\gamma + S\alpha\beta S\beta\gamma} = \&c.$$

The interpretation of each of these forms gives a different theorem in spherical trigonometry.

114. A curious proposition, due to Hamilton, gives us a quaternion expression for the *spherical excess* in any triangle. The following proof, which is very nearly the same as one of his, though by no means the simplest that can be given, is chosen here because it incidentally gives a good deal of other information. We leave the quaternion proof as an exercise.

Let the unit-vectors drawn from the centre of the sphere to *A, B, C,* respectively, be α, β, γ. It is required to express, as an arc and as an angle on the sphere, the quaternion

$$\beta\alpha^{-1}\gamma.$$



The figure represents an orthographic projection made on a plane perpendicular to γ. Hence *C* is the centre of the circle *DEe*. Let the great circle through *A, B* meet *DEe* in *E, e,* and let *DE* be a quadrant. Thus $\overset{\frown}{DE}$ represents γ (§ 72). Also make $\overset{\frown}{EF} = \overset{\frown}{AB} = \beta\alpha^{-1}$. Then, evidently,

$$\overset{\frown}{DF} = \beta\alpha^{-1}\gamma,$$

which gives the arcual representation required.

Let DF cut Ee in G. Make $Ca = EG$, and join D, a, and a, F. Obviously, as D is the pole of Ee, Da is a quadrant; and since $EG = Ca$, $Ga = EC$, a quadrant also. Hence a is the pole of DG, and therefore the quaternion may be represented by the angle DaF.

Make $Cb = Ca$, and draw the arcs $Pa\beta$, Pba from P, the pole of AB. Comparing the triangles Eba and $ea\beta$, we see that $Ea = e\beta$. But, since P is the pole of AB, $F\beta a$ is a right angle: and therefore as Fa is a quadrant, so is $F\beta$. Thus AB is the complement of Ea or βe, and therefore

$$a\beta = 2\,AB.$$

Join bA and produce it to c so that $Ac = bA$; join c, P, cutting AB in o. Also join c, B, and B, a.

Since P is the pole of AB, the angles at o are right angles; and therefore, by the equal triangles baA, coA, we have

$$aA = Ao.$$

But
$$a\beta = 2AB,$$
whence
$$oB = B\beta,$$

and therefore the triangles coB and $Ba\beta$ are equal, and c, B, a lie on the same great circle.

Produce cA and cB to meet in H (on the opposite side of the sphere). H and c are diametrically opposite, and therefore cP, produced, passes through H.

Now $Pa = Pb = PH$, for they differ from quadrants by the equal arcs $a\beta$, ba, oc. Hence these arcs divide the triangle Hab into three isosceles triangles.

But
$$\angle PHb + \angle PHa = \angle aHb = \angle bca.$$

Also
$$\angle Pab = \pi - \angle cab - \angle PaH,$$
$$\angle Pba = \angle Pab = \pi - \angle cba - \angle PbH.$$

Adding,
$$2\angle Pab = 2\pi - \angle cab - \angle cba - \angle bca$$
$$= \pi - (\text{spherical excess of } abc).$$

But, as $\angle Fa\beta$ and $\angle Dae$ are right angles, we have

$$\text{angle of } \beta\alpha^{-1}\gamma = \angle FaD = \angle \beta ae = \angle Pab$$
$$= \frac{\pi}{2} - \tfrac{1}{2}\,(\text{spherical excess of } abc).$$

[Numerous singular geometrical theorems, easily proved *ab initio* by quaternions, follow from this: e. g. The arc *AB*, which bisects two sides of a spherical triangle *abc*, intersects the base at the distance of a quadrant from its middle point. All spherical triangles, with a common side, and having their other sides bisected by the same great circle (i. e. having their vertices in a small circle parallel to this great circle) have equal areas, &c., &c.]

115. Let $\overline{Oa} = \alpha'$, $\overline{Ob} = \beta'$, $\overline{Oc} = \gamma'$, and we have

$$\left(\frac{\alpha'}{\beta'}\right)^{\frac{1}{2}} \left(\frac{\beta'}{\gamma'}\right)^{\frac{1}{2}} \left(\frac{\gamma'}{\alpha'}\right)^{\frac{1}{2}} = \frown Ca . \frown cA . \frown Bc$$
$$= \frown Ca . \frown BA$$
$$= \frown EG . \frown FE = \frown FG.$$

But *FG* is the complement of *DF*. Hence the *angle of the quaternion*

$$\left(\frac{\alpha'}{\beta'}\right)^{\frac{1}{2}} \left(\frac{\beta'}{\gamma'}\right)^{\frac{1}{2}} \left(\frac{\gamma'}{\alpha'}\right)^{\frac{1}{2}}$$

is half the spherical excess of the triangle whose angular points are at the extremities of the unit-vectors α', β', γ'.

[In seeking a purely quaternion proof of the preceding propositions, the student may commence by showing that for any three unit-vectors we have

$$\frac{\beta}{\alpha}\,\frac{\gamma}{\beta}\,\frac{\alpha}{\gamma} = -(\beta\alpha^{-1}\gamma)^2.$$

The angle of the first of these quaternions can be easily assigned; and the equation shows how to find that of $\beta\alpha^{-1}\gamma$. But a still simpler method of proof is easily derived from the composition of rotations.]

116. A *scalar* equation in ρ, the vector of an undetermined point, is generally the equation of a *surface;* since we may substitute for ρ the expression

$$\rho = x\alpha,$$

where x is an unknown scalar, and α any assumed unit-vector. The result is an equation to determine x. Thus one or more points are found on the vector $x\alpha$ whose coördinates satisfy the equation; and the locus is a surface whose degree is determined by that of the equation which gives the values of x.

But a *vector* equation in ρ, as we have seen, generally leads to three scalar equations, from which the three rectangular or other components of the sought vector are to be derived. Such a vector equation, then, usually belongs to a definite number of *points* in space. But in certain cases these may form a *line,* and even a *surface,* the vector equation losing as it were one or two of the three scalar equations to which it is usually equivalent.

Thus while the equation

$$\alpha\rho = \beta$$

gives at once

$$\rho = \alpha^{-1}\beta,$$

which is the vector of a definite point (since we have evidently

$$S\alpha\beta = 0);$$

the closely allied equation

$$V\alpha\rho = \beta$$

is easily seen to involve

$$S\alpha\beta = 0,$$

and to be satisfied by

$$\rho = \alpha^{-1}\beta + x\alpha,$$

whatever be x. Hence the vector of any point whatever in the line drawn parallel to α from the extremity of $\alpha^{-1}\beta$ satisfies the given equation.

117. Again, $V\alpha\rho . V\rho\beta = (V\alpha\beta)^2$

is equivalent to but two scalar equations. For it shews that

$V\alpha\rho$ and $V\beta\rho$ are parallel, i. e. ρ lies in the same plane as α and β, and can therefore be written (§ 24)

$$\rho = x\alpha + y\beta,$$

where x and y are scalars as yet undetermined.

We have now

$$V\alpha\rho = yV\alpha\beta,$$

$$V\rho\beta = xV\alpha\beta,$$

which, by the given equation, lead to

$$xy = 1, \quad \text{or} \quad y = \frac{1}{x}, \quad \text{or finally}$$

$$\rho = x\alpha + \frac{1}{x}\beta;$$

which (§ 40) is the equation of a hyperbola whose asymptotes are in the directions of α and β.

118. Again, the equation

$$V.V\alpha\beta V\alpha\rho = 0,$$

though apparently equivalent to three scalar equations, is really equivalent to one only. In fact we see by § 91 that it may be written

$$-\alpha S.\alpha\beta\rho = 0,$$

whence, if α be not zero, we have

$$S.\alpha\beta\rho = 0,$$

and thus (§ 101) the only condition is that ρ is coplanar with α, β. Hence the equation represents the plane in which α and β lie.

119. Some very curious results are obtained when we extend these processes of interpretation to functions of a *quaternion*

$$q = \omega + \rho$$

instead of functions of a mere *vector* ρ.

A scalar equation containing such a quaternion, along with quaternion constants, gives, as in last section, the equation of a surface, if we assign a definite value to ω. Hence for successive values of ω, we have successive surfaces belonging to a system; and thus when ω is indeterminate the equation represents not a *surface*, as before, but a *volume*, in the sense that the vector of any point within that volume satisfies the equation.

Thus the equation

$$(Tq)^2=a^2,$$

or

$$\omega^2-\rho^2=a^2,$$

or

$$(T\rho)^2 = a^2-\omega^2,$$

represents, for any assigned value of ω, not greater than a, a sphere whose radius is $\sqrt{a^2-\omega^2}$. Hence the equation is satisfied by the vector of any point whatever in the *volume* of a sphere of radius a, whose centre is origin.

Again, by the same kind of investigation,

$$(T(q-\beta))^2=a^2,$$

where $q=\omega+\rho$ is easily seen to represent the volume of a sphere of radius a described about the extremity of β as centre.

Also $S(q^2)=-a^2$ is the equation of infinite space less the space contained in a sphere of radius a about the origin.

Similar consequences as to the interpretation of vector equations in quaternions may be readily deduced by the reader.

120. The following transformation is enunciated by Hamilton (*Lectures*, p. 587, and *Elements*, p. 299).

$$r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1} = U(rq+KrKq).$$

Let $r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1} = t$, then

$Tt = 1$, and therefore

$Kt = t^{-1}$;

But $(r^2q^2)^{\frac{1}{2}} = rtq$,

or $r^2q^2 = rtqrtq$,

or $rq = tqrt$.

Hence $KqKr = t^{-1}KrKqt^{-1}$,

or $KrKq = tKqKrt$.

Thus we have

$$U(rq \pm KrKq) = tU(qr \pm KqKr)t,$$

or, if we put $$s = U(qr \pm KqKr),$$

$$Ks = \pm tst.$$

Hence $$sKs = (Ts)^2 = 1 = \pm stst,$$

which, if we take the positive sign, requires

$$st = \pm 1,$$

or $$t = \pm s^{-1} = \pm UKs,$$

which is the required transformation.

[It is to be noticed that there are other results which might have been arrived at by using the negative sign above; some involving an arbitrary unit-vector, others involving the imaginary of ordinary algebra.]

121. As a final example, we take a transformation of Hamilton's, of great importance in the theory of surfaces of the second order.

Transform the expression

$$(S\alpha\rho)^2 + (S\beta\rho)^2 + (S\gamma\rho)^2$$

in which α, β, γ are any three mutually rectangular vectors, into the form

$$\left(\frac{T(\iota\rho + \rho\kappa)}{\kappa^2 - \iota^2}\right)^2,$$

which involves only two vector-constants, ι, κ.

$$\begin{aligned}\{T(\iota\rho + \rho\kappa)\}^2 &= (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) \quad (\S\S\ 52,\ 55)\\ &= (\iota^2 + \kappa^2)\rho^2 + (\iota\rho\kappa\rho + \rho\kappa\rho\iota)\\ &= (\iota^2 + \kappa^2)\rho^2 + 2\,S.\iota\rho\kappa\rho\\ &= (\iota - \kappa)^2\rho^2 + 4\,S\iota\rho S\kappa\rho.\end{aligned}$$

Hence $$(S\alpha\rho)^2 + (S\beta\rho)^2 + (S\gamma\rho)^2 = \frac{(\iota - \kappa)^2}{(\kappa^2 - \iota^2)^2}\,\rho^2 + 4\,\frac{S\iota\rho S\kappa\rho}{(\kappa^2 - \iota^2)^2}.$$

But $$\alpha^{-2}(S\alpha\rho)^2 + \beta^{-2}(S\beta\rho)^2 + \gamma^{-2}(S\gamma\rho)^2 = \rho^2 \quad (\S\S\ 25,\ 73).$$

Multiply by β^2 and subtract, we get

$$\left(1-\frac{\beta^2}{\alpha^2}\right)(S\alpha\rho)^2-\left(\frac{\beta^2}{\gamma^2}-1\right)(S\gamma\rho)^2=\left\{\frac{(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2}-\beta^2\right\}\rho^2+4\frac{S\iota\rho S\kappa\rho}{(\kappa^2-\iota^2)^2}.$$

The left side breaks up into two real factors if β^2 be intermediate in value to α^2 and γ^2: and that the right side may do so the term in ρ^2 must vanish. This condition gives

$$\beta^2=\frac{(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2};$$ and the identity becomes

$$S\left(\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}+\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right)\rho S\left(\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}-\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right)\rho$$

$$=4\frac{S\iota\rho S\kappa\rho}{(\kappa^2-\iota^2)^2}.$$

Hence we must have

$$\frac{2\iota}{\kappa^2-\iota^2}=p\left(\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}+\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right),$$

$$\frac{2\kappa}{\kappa^2-\iota^2}=\frac{1}{p}\left(\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}-\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right),$$

where p is an undetermined scalar.

To determine p, substitute in the expression for β^2, and we find

$$4\beta^2=\frac{4(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2}=\left(p-\frac{1}{p}\right)^2(\alpha^2-\beta^2)+\left(p+\frac{1}{p}\right)^2(\beta^2-\gamma^2)$$

$$=\left(p^2+\frac{1}{p^2}\right)(\alpha^2-\gamma^2)-2(\alpha^2+\gamma^2)+4\beta^2.$$

Thus the transformation succeeds if

$$p^2+\frac{1}{p^2}=\frac{2(\alpha^2+\gamma^2)}{\alpha^2-\gamma^2},$$

which gives
$$p+\frac{1}{p}=\pm 2\sqrt{\frac{\alpha^2}{\alpha^2-\gamma^2}},$$

$$p-\frac{1}{p}=\pm 2\sqrt{\frac{\gamma^2}{\alpha^2-\gamma^2}}.$$

Hence
$$\frac{4(\kappa^2-\iota^2)}{(\kappa^2-\iota^2)^2}=\left(\frac{1}{p^2}-p^2\right)(\alpha^2-\gamma^2)=\pm 4\sqrt{\alpha^2\gamma^2},$$

or
$$(\kappa^2-\iota^2)^{-1}=\pm T\alpha T\gamma.$$

Again, $$p = \frac{T\alpha + T\gamma}{\sqrt{\gamma^2 - \alpha^2}}, \quad \frac{1}{p} = \frac{T\alpha - T\gamma}{\sqrt{\gamma^2 - \alpha^2}},$$
and therefore
$$2\iota = \frac{T\alpha + T\gamma}{T\alpha\, T\gamma}\left(\sqrt{\frac{\beta^2 - \alpha^2}{\gamma^2 - \alpha^2}}\, U\alpha + \sqrt{\frac{\gamma^2 - \beta^2}{\gamma^2 - \alpha^2}}\, U\gamma\right),$$
$$2\kappa = \frac{T\alpha - T\gamma}{T\alpha\, T\gamma}\left(\sqrt{\frac{\beta^2 - \alpha^2}{\gamma^2 - \alpha^2}}\, U\alpha - \sqrt{\frac{\gamma^2 - \beta^2}{\gamma^2 - \alpha^2}}\, U\gamma\right).$$
Thus we have proved the possibility of the transformation, and determined the transforming vectors ι, κ.

122. By differentiating the equation
$$(S\alpha\rho)^2 + (S\beta\rho)^2 + (S\gamma\rho)^2 = \left(\frac{T(\iota\rho + \rho\kappa)}{(\kappa^2 - \iota^2)}\right)^2$$
we obtain, as will be seen in Chapter IV, the following,
$$S\alpha\rho S\alpha\rho' + S\beta\rho S\beta\rho' + S\gamma\rho S\gamma\rho' = \frac{S.(\iota\rho + \rho\kappa)(\kappa\rho' + \rho'\iota)}{(\kappa^2 - \iota^2)^2},$$
where ρ' also may be any vector whatever.

This is another very important formula of transformation; and it will be a good exercise for the student to prove its truth by processes analogous to those in last section. We may merely observe, what indeed is obvious, that by putting $\rho' = \rho$ it becomes the formula of last section. And we see that we may write, with the recent values of ι and κ in terms of α, β, γ, the identity
$$\alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho = \frac{(\iota^2 + \kappa^2)\rho + 2V.\iota\rho\kappa}{(\kappa^2 - \iota^2)^2}$$
$$= \frac{(\iota - \kappa)^2\rho + 2(\iota S\kappa\rho + \kappa S\iota\rho)}{(\kappa^2 - \iota^2)^2}.$$

123. In various quaternion investigations, especially in such as involve *imaginary* intersections of curves and surfaces, the old imaginary of algebra of course appears. But it is to be particularly noticed that this expression is analogous to a scalar and not to a vector, and that like real scalars it is commutative in multiplication with all other factors. Thus it appears, by the same proof as in algebra, that any quaternion expression which contains this imaginary can always be broken up into the sum of two parts, one real, the other multiplied by the first power of $\sqrt{-1}$. Such an expression, viz.

$$q = q' + \sqrt{-1}\,q'',$$

where q' and q'' are real quaternions, is called a BIQUATERNION. Some little care is requisite in the management of these expressions, but there is no new difficulty. The points to be observed are: first, that any biquaternion can be divided into a real and imaginary part, the latter being the product of $\sqrt{-1}$ by a real quaternion; second, that this $\sqrt{-1}$ is commutative with all other quantities in multiplication; third, that if two biquaternions be equal, as

$$q' + \sqrt{-1}\,q'' = r' + \sqrt{-1}\,r'',$$

we have, as in algebra,

$$q' = r', \qquad q'' = r'';$$

so that an equation between biquaternions involves in general *eight* equations between scalars. Compare § 80.

124. We have, obviously, since $\sqrt{-1}$ is a scalar,

$$S(q' + \sqrt{-1}\,q'') = Sq' + \sqrt{-1}\,Sq'',$$

$$V(q' + \sqrt{-1}\,q'') = Vq' + \sqrt{-1}\,Vq''.$$

Hence (§ 103)

$$\begin{aligned}\{T(q' + \sqrt{-1}\,q'')\}^2 &= (Sq')^2 - (Sq'')^2 - (Vq')^2 + (Vq'')^2 \\ &\qquad + 2\sqrt{-1}\,\{Sq'Sq'' - S.Vq'Vq''\}, \\ &= (Tq')^2 - (Tq'')^2 + 2\sqrt{-1}\,S.q'Kq''.\end{aligned}$$

The only remark which need be made on such formulæ is this, that *the tensor of a biquaternion may vanish while both of the component quaternions are finite.*

Thus, if $$Tq' = Tq'',$$

and $$S.q'Kq'' = 0,$$

the above formula gives

$$T(q' + \sqrt{-1}\,q'') = 0.$$

The condition $\qquad S.q'Kq'' = 0$

may be written

$$Kq'' = q'^{-1}a, \quad \text{or} \quad q'' = -aKq'^{-1} = -\frac{aq'}{(Tq')^2},$$

where a is an indeterminate vector.

Hence $\qquad Tq' = Tq'' = TKq'' = \dfrac{Ta}{Tq'},$

and therefore

$$Tq'(Uq' - \sqrt{-1}\,Ua.Uq')$$

is the general form of a biquaternion whose tensor is zero.

125. More generally we have, q, r, q', r' being any four real and non-evanescent quaternions,

$$(q + \sqrt{-1}\,q')(r + \sqrt{-1}\,r') = qr - q'r' + \sqrt{-1}(qr' + q'r).$$

That this product may vanish we must have

$$qr = q'r',$$

and $\qquad qr' = -q'r.$

Eliminating r' we have

$$qq'^{-1}qr = -q'r,$$

which gives $\qquad (q'^{-1}q)^2 = -1,$

i. e. $\qquad q = q'a$

where a is some unit-vector.

And the two equations now agree in giving

$$-r = ar',$$

so that we have the biquaternion factors in the form

$$q'(a + \sqrt{-1}) \quad \text{and} \quad -(a - \sqrt{-1})r';$$

and their product is

$$-q'(a + \sqrt{-1})(a - \sqrt{-1})r',$$

which, of course, vanishes.

[A somewhat simpler investigation of the same proposition may be obtained by writing the biquaternions as

$$q'(q'^{-1}q+\sqrt{-1}) \quad \text{and} \quad (rr'^{-1}+\sqrt{-1})r',$$

$$\text{or} \qquad q'(q''+\sqrt{-1}) \quad \text{and} \quad (r''+\sqrt{-1})r',$$

and showing that

$$q''=-r''=\alpha, \text{ where } T\alpha=1.]$$

From this it appears that if the product of two *bivectors*

$$\rho+\sigma\sqrt{-1} \quad \text{and} \quad \rho'+\sigma'\sqrt{-1}$$

is zero, we must have

$$\sigma^{-1}\rho=-\rho'\sigma'^{-1}=U\alpha,$$

where α may be any vector whatever. But this result is still more easily obtained by means of a direct process.

126. It may be well to observe here (as we intend to avail ourselves of it in the succeeding Chapters) that certain abbreviated forms of expression may be used when they are not liable to confuse, or lead to error. Thus we may write

$$T^2q \text{ for } (Tq)^2,$$

just as we write

$$\cos^2\theta \text{ for } (\cos\theta)^2,$$

although the true meanings of these expressions are

$$T(T\alpha) \text{ and } \cos(\cos\theta).$$

The former is justifiable, as $T(T\alpha)=T\alpha$, and therefore $T^2\alpha$ is not required to signify the second tensor (or tensor of the tensor) of α. But the trigonometrical usage is quite indefensible.

Similarly we may write

$$S^2q \text{ for } (Sq)^2, \text{ \&c.}$$

but it may be advisable not to use

$$Sq^2$$

as the equivalent of either of those just written; inasmuch as it might be confounded with the (generally) different quantity

$$S.q^2 \text{ or } S(q^2),$$

although this is rarely written without the point or the brackets.

127. The beginner may expect to be a little puzzled with this aspect of the notation at first; but, as he learns more of the subject, he will soon see clearly the distinction between such an expression as

$$S.V\alpha\beta V\beta\gamma,$$

where we may omit at pleasure either the point or the first V without altering the value, and the very different one

$$S\alpha\beta.V\beta\gamma,$$

which admits of no such changes, without altering its value.

All these simplifications of notation are, in fact, merely examples of the transformations of quaternion expressions to which part of this Chapter has been devoted. Thus, to take a very simple example, we easily see that

$$\begin{aligned} S.V\alpha\beta V\beta\gamma &= SV\alpha\beta V\beta\gamma = S.\alpha\beta V\beta\gamma = S\alpha V.\beta V\beta\gamma = -S\alpha V.(V\beta\gamma)\beta \\ &= S\alpha V.(V\gamma\beta)\beta = S.\alpha V(\gamma\beta)\beta = S.V(\gamma\beta)\beta\alpha = SV\gamma\beta V\beta\alpha \\ &= S.\gamma\beta V\beta\alpha = \&c., \&c. \end{aligned}$$

The above group does not nearly exhaust the list of even the simpler ways of expressing the given quantity. We recommend it to the careful study of the reader.

EXAMPLES TO CHAPTER III.

1. Investigate, by quaternions, the requisite formulæ for changing from any set of cöordinate axes to another; and derive from your general result, and also from special investigations, the common expressions for the following cases :—

(*a*.) Rectangular axes turned about z through any angle.

(*b.*) Rectangular axes turned into any new position by rotation about a line equally inclined to the three.

(*c.*) Rectangular turned to oblique, one of the new axes lying in each of the former cöordinate planes.

2. If $T\rho = T\alpha = T\beta = 1$, and $S.\alpha\beta\rho = 0$, show by direct transformations that

$$S.U(\rho-\alpha)U(\rho-\beta) = \sqrt{\tfrac{1}{2}(1-S\alpha\beta)}.$$

Interpret this theorem geometrically.

3. If $S\alpha\beta = 0$, $T\alpha = T\beta = 1$, show that

$$(1+\alpha^m)\beta = 2\cos\frac{m\pi}{4}\alpha^{\frac{m}{2}}\beta = 2\,S\alpha^{\frac{m}{2}}.\alpha^{\frac{m}{2}}\beta.$$

4. Put in its simplest form the equation

$$\rho S.V\alpha\beta V\beta\gamma V\gamma\alpha = aV.V\gamma\alpha V\alpha\beta + bV.V\alpha\beta V\beta\gamma + cV.V\beta\gamma V\gamma\alpha;$$

and show that

$$a = S.\beta\gamma\rho, \text{ \&c.}$$

5. Prove the following theorems, and exhibit them as properties of determinants:—

(*a.*) $S.(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = 2S.\alpha\beta\gamma$.

(*b.*) $S.V\alpha\beta V\beta\gamma V\gamma\alpha = -(S.\alpha\beta\gamma)^2$.

(*c.*) $S.V(\alpha+\beta)(\beta+\gamma)V(\beta+\gamma)(\gamma+\alpha)V(\gamma+\alpha)(\alpha+\beta) = -4(S.\alpha\beta\gamma)^2$.

(*d.*) $S.V(V\alpha\beta V\beta\gamma)V(V\beta\gamma V\gamma\alpha)V(V\gamma\alpha V\alpha\beta) = -(S.\alpha\beta\gamma)^4$.

(*e.*) $S.\delta\epsilon\zeta = 16(S.\alpha\beta\gamma)^4$, where

$$\delta = V(V(\alpha+\beta)(\beta+\gamma)V(\beta+\gamma)(\gamma+\alpha)),$$
$$\epsilon = V(V(\beta+\gamma)(\gamma+\alpha)V(\gamma+\alpha)(\alpha+\beta)),$$
$$\zeta = V(V(\gamma+\alpha)(\alpha+\beta)V(\alpha+\beta)(\beta+\gamma)).$$

6. Prove the common formula for the product of two determinants of the third order in the form

$$S.\alpha\beta\gamma S.\alpha_1\beta_1\gamma_1 = \begin{vmatrix} S\alpha\alpha_1 & S\beta\alpha_1 & S\gamma\alpha_1 \\ S\alpha\beta_1 & S\beta\beta_1 & S\gamma\beta_1 \\ S\alpha\gamma_1 & S\beta\gamma_1 & S\gamma\gamma_1 \end{vmatrix}.$$

7. The lines bisecting pairs of opposite sides of a quadrilateral are perpendicular to each other when the diagonals of the quadrilateral are equal.

8. Show that

(*a*.) $S.q^2 = 2S^2q - T^2q$,

(*b*.) $S.q^3 = S^3q - 3\,SqT^2Vq$,

(*c*.) $\alpha^2\beta^2\gamma^2 + S^2.\alpha\beta\gamma = V^2.\alpha\beta\gamma$,

(*d*.) $S(V.\alpha\beta\gamma V.\beta\gamma\alpha V.\gamma\alpha\beta) = 4\,S\alpha\beta S\beta\gamma S\gamma\alpha S.\alpha\beta\gamma$,

(*e*.) $V.q^3 = (3\,S^2q - T^2Vq)Vq$,

(*f*.) $qUVq^{-1} = Sq.UVq - TVq$;

and interpret each as a formula in plane or spherical trigonometry.

9. If q be an undetermined quaternion, what loci are represented by

(*a*.) $(q\alpha^{-1})^2 = -a^2$,

(*b*.) $(q\alpha^{-1})^4 = a^4$,

(*c*.) $S.(q-\alpha)^2 = a^2$,

where a is any given scalar and α any given vector?

10. If q be any quaternion, show that the equation

$$Q^2 = q^2$$

is satisfied, not alone by $Q = \pm q$ but also, by

$$Q = \pm\sqrt{-1}\,(Sq.UVq - TVq).$$

(Hamilton, *Lectures*, p. 673.)

11. Wherein consists the difference between the two equations

$$T^2\frac{\rho}{\alpha} = 1, \quad \text{and} \quad \left(\frac{\rho}{\alpha}\right)^2 = -1\,?$$

What is the full interpretation of each, α being a given, and ρ an undetermined, vector?

12. Find the full consequences of each of the following groups of equations, both as regards the unknown vector ρ and the given vectors α, β, γ.

$$(a.)\ \begin{array}{l} S.\alpha\beta\rho = 0, \\ S.\beta\gamma\rho = 0, \end{array} \qquad (b.)\ \begin{array}{l} S\alpha\rho \ \ = 0, \\ S.\alpha\beta\rho = 0, \\ S\beta\rho \ \ = 0, \end{array} \qquad (c.)\ \begin{array}{l} S\alpha\rho \ \ \ = 0, \\ S.\alpha\beta\rho \ = 0, \\ S.\alpha\beta\gamma\rho = 0. \end{array}$$

13. From §§ 74, 109, show that, if ϵ be any unit-vector, and m any scalar, $\epsilon^m = \cos\frac{m\pi}{2} + \epsilon\sin\frac{m\pi}{2}$.

Hence show that if α, β, γ be radii drawn to the corners of a triangle on the unit-sphere, whose spherical excess is m right angles,

$$\frac{\alpha+\beta}{\beta+\gamma}\cdot\frac{\gamma+\alpha}{\alpha+\beta}\cdot\frac{\beta+\gamma}{\gamma+\alpha} = \alpha^m.$$

Also that, if A, B, C be the angles of the triangle, we have

$$\gamma^{\frac{2C}{\pi}}\beta^{\frac{2B}{\pi}}\alpha^{\frac{2A}{\pi}} = -1.$$

14. Show that for *any* three vectors α, β, γ, we have

$$(U\alpha\beta)^2 + (U\beta\gamma)^2 + (U\alpha\gamma)^2 + (U.\alpha\beta\gamma)^2 + 4\,U\alpha\gamma.SU\alpha\beta SU\beta\gamma = -2.$$

(Hamilton, *Elements*, p. 388.)

15. If a_1, a_2, a_3, x, be any four scalars, and ρ_1, ρ_2, ρ_3 any three vectors, show that

$$\begin{aligned}(S.\rho_1\rho_2\rho_3)^2 + (\Sigma.a_1V\rho_2\rho_3)^2 + x^2(\Sigma V\rho_1\rho_2)^2 - x^2(\Sigma.a_1(\rho_2-\rho_3))^2 \\ + 2\Pi(x^2 + S\rho_1\rho_2 + a_1a_2) = 2\Pi(x^2+a_1^2) + 2\Pi a^2 \\ + \Sigma\{(x^2+a_1^2+\rho_1^2)((V\rho_2\rho_3)^2 + 2a_2a_3(x^2+S\rho_2\rho_3) - x^2(\rho_2-\rho_3)^2)\};\end{aligned}$$

$$\text{where} \qquad \Pi a^2 = a_1^2a_2^2a_3^2.$$

Verify this formula by a simple process in the particular case

$$a_1 = a_2 = a_3 = x = 0.$$

(*Ibid.*)

CHAPTER IV.

DIFFERENTIATION OF QUATERNIONS.

128. IN Chapter I we have already considered as a special case the differentiation of a *vector* function of a scalar independent variable: and it is easy to see at once that a similar process is applicable to a *quaternion* function of a scalar independent variable. The differential, or differential coefficient, thus found, is in general another function of the same scalar variable; and can therefore be differentiated anew by a second, third, &c. application of the same process. And precisely similar remarks apply to partial differentiation of a quaternion function of any number of *scalar* independent variables. In fact, this process is identical with ordinary differentiation.

129. But when we come to differentiate a function of a vector, or of a quaternion, some caution is requisite; there is, in general, nothing which can be called a differential coefficient; and in fact we require (as already hinted in § 36) to employ a definition of a differential, somewhat different from the ordinary one, but coinciding with it when applied to functions of mere scalar variables.

130. If $r = F(q)$ be a function of a quaternion q,

$$dr = dFq = \mathcal{L}_{\infty} n \left\{ F\left(q + \frac{dq}{n}\right) - F(q) \right\},$$

where n is a scalar which is ultimately to be made infinite, is *defined* to be the differential of r or Fq.

Here dq may be *any quaternion whatever*, and the right-hand member may be written

$$f(q, dq),$$

where f is a new function, depending on the form of F; homogeneous and of the *first* degree in dq; but not, in general, capable of being put in the form

$$\mathrm{f}(q)\,dq.$$

131. To make more clear these last remarks, we may observe that the function

$$f(q, dq),$$

thus derived as the differential of $F(q)$, is *distributive* with respect to dq. That is

$$f(q, r+s)=f(q, r)+f(q, s),$$

r and s being any quaternions.

For $$f(q, r+s)=\mathcal{L}_\infty n\left(F\left(q+\frac{r+s}{n}\right)-F(q)\right)$$

$$=\mathcal{L}_\infty n\left\{F\left(q+\frac{r}{n}+\frac{s}{n}\right)-F\left(q+\frac{s}{n}\right)+F\left(q+\frac{s}{n}\right)-Fq\right\}$$

$$=\mathcal{L}_\infty n\left\{F\left(q+\frac{s}{n}+\frac{r}{n}\right)-F\left(q+\frac{s}{n}\right)\right\}+\mathcal{L}_\infty n\left\{F\left(q+\frac{s}{n}\right)-Fq\right\}$$

$$=f(q, r)+f(q, s).$$

And, as a particular case, it is obvious that if x be any scalar

$$f(q, xr)=xf(q, r).$$

132. And if we define in the same way

$dF(q, r, s \ldots\ldots)$ as being the value of

$$\mathcal{L}_\infty n\left\{F\left(q+\frac{dq}{n}, r+\frac{dr}{n}, s+\frac{ds}{n}, \ldots\ldots\right)-F(q, r, s, \ldots\ldots)\right\},$$

where $q, r, s, \ldots\, dq, dr, ds, \ldots\ldots$ are any quaternions whatever; we shall obviously arrive at a result which may be written

$$f(q, r, s, \ldots\, dq, dr, ds, \ldots\ldots),$$

where f is homogeneous and linear in the system of quaternions $dq, dr, ds, \ldots\ldots$ and distributive with respect to each of them. Thus, in differentiating any power, product, &c. of one or more quaternions, each factor is to be differentiated as if it alone were variable; and the terms corresponding to these are to be added for the complete differential. This differs from the ordinary process of scalar differentiation solely in the fact that, on account of the non-commutative property of quaternion multiplication, each factor must be differentiated *in situ*. Thus

$$d(qr) = dq.r + qdr, \text{ but } not \text{ generally } = rdq + qdr.$$

133. As Examples we take chiefly those which lead to results of constant use to us in succeeding Chapters. Some of the work will be given at full length as an exercise in quaternion transformations.

$$(1) \quad (T\rho)^2 = -\rho^2.$$

The differential of the left-hand side is simply, since $T\rho$ is a scalar,

$$2\,T\rho\,dT\rho.$$

That of ρ^2 is

$$\mathcal{L}_\infty n \left(\left(\rho + \frac{d\rho}{n}\right)^2 - \rho^2\right)$$

$$= \mathcal{L}_\infty n \left(\frac{2}{n} S\rho d\rho + \frac{(d\rho)^2}{n^2}\right) \ (\S\ 104)$$

$$= 2\,S\rho d\rho.$$

Hence $$T\rho\,dT\rho = -S\rho d\rho,$$

or $$dT\rho = -S.U\rho\,d\rho = S\frac{d\rho}{U\rho},$$

or $$\frac{dT\rho}{T\rho} = S\frac{d\rho}{\rho}.$$

(2) Again,

$$\rho = T\rho\,U\rho$$

$$d\rho = dT\rho.U\rho + T\rho\,dU\rho,$$

whence
$$\frac{d\rho}{\rho} = \frac{dT\rho}{T\rho} + \frac{dU\rho}{U\rho}$$
$$= S\frac{d\rho}{\rho} + \frac{dU\rho}{U\rho} \quad \text{by (1).}$$

Hence
$$\frac{dU\rho}{U\rho} = V\frac{d\rho}{\rho}.$$

This may be transformed into
$$V\frac{d\rho\,\rho}{\rho^2} \text{ or } \frac{V\rho\, d\rho}{T\rho^2}, \text{ \&c.}$$

(3) $(Tq)^2 = qKq$

$$2Tq\,dTq = d(qKq) = \mathcal{L}_{\infty}\, n\left[\left(q + \frac{dq}{n}\right) K\left(q + \frac{dq}{n}\right) - qKq\right],$$
$$= \mathcal{L}_{\infty}\, n\left(\frac{qKdq + dqKq}{n} + \frac{1}{n^2}\, dq\, Kdq\right),$$
$$= qKdq + dqKq,$$
$$= qKdq + K(qKdq),$$
$$= 2S.qKdq = 2S.Kq\,dq.$$

Hence
$$dTq = S.UKq\,dq = S.Uq^{-1}dq$$

since
$$Tq = TKq, \quad \text{and} \quad UKq = Uq^{-1}.$$

If $q = \rho$, a vector, $Kq = K\rho = -\rho$, and the formula becomes
$$dT\rho = -S.U\rho\, d\rho \quad \text{as in (1).}$$

Again,
$$\frac{dTq}{Tq} = S\frac{dq}{q}.$$

But
$$dq = Tq\,dUq + Uq\,dTq,$$

which gives
$$\frac{dq}{q} = \frac{dTq}{Tq} + \frac{dUq}{Uq};$$

whence, as
$$S\frac{dq}{q} = \frac{dTq}{Tq},$$

we have
$$V\frac{dq}{q} = \frac{dUq}{Uq}.$$

(4) $$d(q^2) = \mathcal{L}_{\infty}\, n \left(\left(q + \frac{dq}{n}\right)^2 - q^2\right)$$
$$= qdq + dqq$$
$$= 2S.qdq + 2Sq.Vdq + 2Sdq.Vq.$$

If q be a vector, as ρ, Sq and Sdq vanish, and we have
$$d(\rho^2) = 2S\rho d\rho \text{ as in (1).}$$

(5) Let $q = r^{\frac{1}{2}}$.

This gives $dr^{\frac{1}{2}} = dq$. But
$$dr = d(q^2) = qdq + dqq.$$
This, multiplied *by* q and *into* Kq, gives
$$qdr = q^2dq + qdqq,$$
and $$drKq = dqTq^2 + qdqKq.$$
Adding, we have
$$qdr + drKq = (q^2 + Tq^2 + 2Sq.q)dq;$$
whence dq, i. e. $dr^{\frac{1}{2}}$, is at once found in terms of dr. This process is given by Hamilton, *Lectures*, p. 628.

(6) $$qq^{-1} = 1,$$
$$qdq^{-1} + dqq^{-1} = 0;$$
$$\therefore\ dq^{-1} = -q^{-1}dqq^{-1}.$$

If q is a vector, $= \rho$ suppose,
$$d\rho^{-1} = -\rho^{-1}d\rho\rho^{-1}$$
$$= \frac{d\rho}{\rho^2} - \frac{2}{\rho}S\frac{d\rho}{\rho}$$
$$= \left(\frac{d\rho}{\rho} - 2S\frac{d\rho}{\rho}\right)\frac{1}{\rho}$$
$$= -K\left(\frac{d\rho}{\rho}\right)\frac{1}{\rho}.$$

(7) $$q = Sq + Vq,$$
$$dq = dSq + dVq.$$
But $$dq = Sdq + Vdq.$$

Comparing, we have

$$dSq = Sdq, \qquad dVq = Vdq.$$

Since $Kq = Sq - Vq$, we find by a similar process

$$dKq = Kdq.$$

134. Successive differentiation of course presents no new difficulty.

Thus, we have seen that

$$d(q^2) = dqq + qdq.$$

Differentiating again, we have

$$d^2(q^2) = d^2q.q + 2(dq)^2 + qd^2q,$$

and so on for higher orders.

If q be a vector, as ρ, we have, § 133 (1),

$$d(\rho^2) = 2S\rho d\rho.$$

Hence $d^2(\rho^2) = 2(d\rho)^2 + 2S\rho d^2\rho$, and so on.

Similarly $d^2U\rho = -d\left(\frac{U\rho}{T\rho^2}V\rho\, d\rho\right)$.

But $$d\frac{1}{T\rho^2} = -\frac{2dT\rho}{T\rho^3} = \frac{2S\rho d\rho}{T\rho^4},$$

and $d.V\rho d\rho = V.\rho d^2\rho$.

Hence

$$-d^2U\rho = \frac{U\rho}{T\rho^4}(V\rho d\rho)^2 + \frac{U\rho V\rho d^2\rho}{T\rho^2} + \frac{2U\rho V\rho d\rho S\rho d\rho}{T\rho^4}$$

$$= \frac{U\rho}{T\rho^4}((V\rho d\rho)^2 - \rho^2 V\rho d^2\rho + 2V\rho d\rho S\rho d\rho).$$

[This may be farther simplified; but it may be well to caution the student that we cannot, for such a purpose, write the above expression as

$$\frac{U\rho}{T\rho^4}V.\rho\,\{d\rho V\rho d\rho - d^2\rho\rho^2 + 2d\rho S\rho d\rho\}.]$$

135. If the first differential of q be considered as a *constant* quaternion, we have, of course,

$$d^2q = 0, \qquad d^3q = 0, \text{ \&c.},$$

and the preceding formulæ become considerably simplified.

Hamilton has shown that in this case *Taylor's Theorem* admits of an easy extension to quaternions. That is, we may write

$$f(q+xdq)=f(q)+xdf(q)+\frac{x^2}{1.2}d^2f(q)+\ldots\ldots$$

if $d^2q=0$; subject, of course, to particular exceptions and limitations as in the ordinary applications to functions of scalar variables. Thus, let

$$\begin{aligned} f(q) &= q^3, \text{ and we have} \\ df(q) &= q^2dq+qdqq+dqq^2, \\ d^2f(q) &= 2dqqdq+2q(dq)^2+2(dq)^2q, \\ d^3f(q) &= 6(dq)^3, \end{aligned}$$

and it is easy to verify by multiplication that we have rigorously

$$(q+xdq)^3=q^3+x(q^2dq+qdqq+dqq^2)+x^2(dqqdq+q(dq)^2+(dq)^2q)+x^3(dq)^3;$$

which is the value given by the application of the above form of Taylor's Theorem.

As we shall not have occasion to employ this theorem, and as the demonstrations which have been found are all too laborious for an elementary treatise, we refer the reader to Hamilton's works, where he will find several of them.

136. To differentiate a function of a function of a quaternion we proceed as with scalar variables, attending to the peculiarities already pointed out.

137. A case of considerable importance in geometrical applications of quaternions is the differentiation of a scalar function of ρ, the vector of any point in space.

Let $$F(\rho)=C,$$

where F is a scalar function and C an arbitrary constant, be the equation of a series of surfaces. Its differential,

$$f(\rho,d\rho)=0,$$

is, of course, a scalar function: and, being homogeneous and linear in $d\rho$, § 130, may be thus written,

$$S\nu d\rho = 0,$$

where ν is a vector, in general a function of ρ.

This vector, ν, is easily seen to have the direction of the *normal* to the given surface at the extremity of ρ; being, in fact, perpendicular to every tangent line $d\rho$, §§ 36, 98. Its length, when F is a surface of the second degree, is the reciprocal of the distance of the tangent-plane from the origin. And we will show, later, that if

$$\rho = ix + jy + kz,$$

then

$$\nu = \left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)F(\rho).$$

EXAMPLES TO CHAPTER IV.

1. Show that

(*a.*) $d.SUq = S.UqV\frac{dq}{q} = -S\frac{dq}{qUVq}TVUq.$

(*b.*) $d.VUq = -V.UqV(dqq^{-1}).$

(*c.*) $d.TVUq = S\frac{dUq}{UVq} = S\frac{dq}{qUVq}SUq.$

(*d.*) $d.\alpha^x = \frac{\pi}{2}\alpha^{x+1}dx.$

(*e.*) $d^2.Tq = \{S^2.dqq^{-1} - S.(dqq^{-1})^2\}Tq = -TqV^2\frac{dq}{q}.$

2. If $F\rho = \Sigma.S\alpha\rho S\beta\rho + \frac{1}{2}g\rho^2$

give $dF\rho = S\nu d\rho,$

show that $\nu = \Sigma V.\alpha\rho\beta + (g + \Sigma S\alpha\beta)\rho.$

CHAPTER V.

THE SOLUTION OF EQUATIONS OF THE FIRST DEGREE.

138. WE have seen that the differentiation of any function whatever of a quaternion, q, leads to an equation of the form

$$dr = f(q, dq),$$

where f is linear and homogeneous in dq. To complete the process of differentiation, we must have the means of solving this equation so as to be able to exhibit directly the value of dq.

This general question is not of so much practical importance as the particular case in which q is a vector; and, besides, as we proceed to show, the solution of the general question may easily be made to depend upon that of the particular case; so that we shall commence with the latter.

The most general expression for the function f is easily seen to be

$$dr = f(q, dq) = \Sigma V.adqb + S.cdq,$$

where a, b, and c may be any quaternion functions of q whatever. Every possible term of a linear and homogeneous function is reducible to this form, as the reader may easily see by writing down all the forms he can devise.

Taking the scalars of both sides, we have

$$Sdr = S.cdq = SdqSc + S.VdqVc.$$

But we have also, by taking the vector parts,

$$Vdr = \Sigma V.adqb = Sdq.\Sigma Vab + \Sigma V.a(Vdq)b.$$

Eliminating Sdq between the equations for Sdr and Vdr it is obvious that a linear and vector expression in Vdq will remain. Such an expression, so far as it contains Vdq, may always be reduced to the form of a sum of terms of the type $\alpha S.\beta Vdq$, by the help of formulæ like those in §§ 90, 91. Solving this, we have Vdq, and Sdq is then found from the preceding equation.

139. The problem may now be stated thus.

Find the value of ρ from the equation

$$\alpha S\beta\rho + \alpha_1 S\beta_1\rho + \ldots = \Sigma.\alpha S\beta\rho = \gamma,$$

where α, β, α_1, β_1, ... γ are given vectors. [It will be shown later that the most general form requires but three terms, i. e. *six* vector constants α, β, α_1, β_1, α_2, β_2 in all.]

If we write, with Hamilton,

$$\phi\rho = \Sigma.\alpha S\beta\rho,$$

the given equation may be written

$$\phi\rho = \gamma,$$

or

$$\rho = \phi^{-1}\gamma,$$

and the object of our investigation is to find the value of the inverse function ϕ^{-1}.

140. We have seen that any vector whatever may be expressed in terms of any three non-coplanar vectors. Hence, we should expect *à priori* that a vector such as $\phi\phi\phi\rho$, or $\phi^3\rho$, for instance, should be capable of expression in terms of ρ, $\phi\rho$, and $\phi^2\rho$. [This is, of course, on the supposition that ρ, $\phi\rho$, and $\phi^2\rho$ are not generally coplanar. But it may easily be seen to extend to this case also. For if these vectors be generally coplanar, so are $\phi\rho$, $\phi^2\rho$, and $\phi^3\rho$, since they may be written σ, $\phi\sigma$, and $\phi^2\sigma$. And thus, of course, $\phi^3\rho$ can be expressed as above. If in a particular case, we should have, for some *definite* vector ρ, $\phi\rho = g\rho$ where g is a scalar, we shall obviously have $\phi^2\rho = g^2\rho$ and $\phi^3\rho = g^3\rho$, so that the equation will still subsist. And a similar explanation holds for the particular case when (for some *definite* value of ρ) ρ, $\phi\rho$, and $\phi^2\rho$ are coplanar. For then we have an equation of the form

$$\phi^2\rho = A\rho + B\phi\rho,$$

which gives

$$\begin{aligned}\phi^3\rho &= A\phi\rho + B\phi^2\rho \\ &= AB\rho + (A+B^2)\phi\rho.\end{aligned}$$

So that $\phi^3\rho$ is in the same plane.]

If, then, we write

$$-\phi^3\rho = x\rho + y\phi\rho + z\phi^2\rho, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

it is evident that x, y, z are quantities independent of the vector ρ, and we can determine them at once by processes such as those in §§ 91, 92.

If any three vectors, as i, j, k, be substituted for ρ, they will in general enable us to assign the values of the three coefficients on the right side of the equation, and the solution is complete. For by putting $\phi^{-1}\rho$ for ρ and transposing, the equation becomes

$$-x\phi^{-1}\rho = y\rho + z\phi\rho + \phi^2\rho;$$

that is, the unknown inverse function is expressed in terms of direct operations. If x vanish, while y remains finite, we substitute $\phi^{-2}\rho$ for ρ, and have

$$-y\phi^{-1}\rho = z\rho + \phi\rho,$$

and if x and y both vanish

$$z\phi^{-1}\rho = \rho.$$

141. To illustrate this process by a simple example we shall take the very important case in which ϕ belongs to a central surface of the second order; suppose an ellipsoid; in which case it will be shown (in Chap. VIII.) that we may write

$$\phi\rho = -a^2iSi\rho - b^2jSj\rho - c^2kSk\rho.$$

Here we have

$$\begin{aligned}\phi i &= a^2 i, & \phi^2 i &= a^4 i, & \phi^3 i &= a^6 i,\\ \phi j &= b^2 j, & \phi^2 j &= b^4 j, & \phi^3 j &= b^6 j,\\ \phi k &= c^2 k, & \phi^2 k &= c^4 k, & \phi^3 k &= c^6 k.\end{aligned}$$

Hence, putting separately i, j, k for ρ in the equation (1) of last section, we have

$$-a^6 = x+ya^2+za^4,$$
$$-b^6 = x+yb^2+zb^4,$$
$$-c^6 = x+yc^2+zc^4.$$

Hence a^2, b^2, c^2 are the roots of the cubic

$$\xi^3+z\xi^2+y\xi+x = 0,$$

which involves the conditions

$$z = -(a^2+b^2+c^2),$$
$$y = a^2b^2+b^2c^2+c^2a^2,$$
$$x = -a^2b^2c^2.$$

Thus, with the above value of ϕ, we have

$$\phi^3\rho = a^2b^2c^2\rho-(a^2b^2+b^2c^2+c^2a^2)\phi\rho+(a^2+b^2+c^2)\phi^2\rho.$$

142. Putting $\phi^{-1}\sigma$ in place of ρ (which is *any* vector whatever) and changing the order of the terms, we have the desired inversion of the function ϕ in the form

$$a^2b^2c^2\phi^{-1}\sigma=(a^2b^2+b^2c^2+c^2a^2)\sigma-(a^2+b^2+c^2)\phi\sigma+\phi^2\sigma,$$

or the inverse function is expressed in terms of the direct function. For this particular case the solution we have given is complete, and satisfactory; and it has the advantage of preparing the reader to expect a similar form of solution in more complex cases.

143. It may also be useful as a preparation for what follows, if we put the equation of § 140 in the form

$$\begin{aligned}0 = \Phi(\rho) &= \phi^3\rho-(a^2+b^2+c^2)\phi^2\rho+(a^2b^2+b^2c^2+c^2a^2)\phi\rho-a^2b^2c^2\rho\\ &= \{\phi^3-(a^2+b^2+c^2)\phi^2+(a^2b^2+b^2c^2+c^2a^2)\phi-a^2b^2c^2\}\rho\\ &= \{(\phi-a^2)(\phi-b^2)(\phi-c^2)\}\rho. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)\end{aligned}$$

This last transformation is permitted because ϕ is commutative with scalars like a^2, i. e. $\phi(a^2\rho) = a^2\phi\rho$.

Here we remark that (by § 140) the equation

$$V.\rho\phi\rho = 0, \quad \text{or} \quad \phi\rho = g\rho,$$

where g is some undetermined scalar, is satisfied, not merely by every vector of null-length, but by the definite system of three rectangular vectors Ai, Bj, Ck whatever be their tensors; the corresponding particular values of g being a^2, b^2, c^2.

144. We now give Hamilton's admirable investigation.

The most general form of a linear and vector function of a vector may of course be written as

$$\phi\rho = \Sigma V.q\rho r,$$

where q and r are any constant quaternions.

Hence, operating by $S.\sigma$ where σ is any other vector,

$$S\sigma\phi\rho = \Sigma S.\sigma V.q\rho r = \Sigma S.\rho V.r\sigma q = S\rho\phi'\sigma, \quad \text{.........} \quad (3)$$

if we agree to write

$$\phi'\sigma = \Sigma V.r\sigma q.$$

The functions ϕ and ϕ' are thus *conjugate* to one another, and on this property the whole investigation depends.

145. Let λ, μ be any two vectors, such that

$$\phi\rho = V\lambda\mu.$$

Operating by $S.\lambda$ and $S.\mu$ we have

$$S\lambda\phi\rho = 0, \qquad S\mu\phi\rho = 0.$$

But, introducing the conjugate function ϕ', these become

$$S\rho\phi'\lambda = 0, \qquad S\rho\phi'\mu = 0,$$

and give ρ in the form

$$m\rho = V\phi'\lambda\phi'\mu,$$

where m is a scalar which, as we shall presently see, is independent of λ, μ, and ρ.

But our original assumption gives

$$\rho = \phi^{-1} V\lambda\mu;$$

hence we have

$$m\phi^{-1} V\lambda\mu = V\phi'\lambda\phi'\mu, \quad \text{.....................} \quad (4)$$

and the problem of inverting ϕ is solved.

146. It remains to find the value of the constant m, and to express the vector

$$V\phi'\lambda\phi'\mu$$

as a function of $V\lambda\mu$.

Operate on (4) by $S.\phi'\nu$, where ν is any vector not coplanar with λ and μ, and we get

$$mS.\phi'\nu\phi^{-1}V\lambda\mu = mS.\nu\phi\phi^{-1}V\lambda\mu \quad \text{(by (3) of § 144)}$$

$$= mS.\lambda\mu\nu = S.\phi'\lambda\phi'\mu\phi'\nu, \quad \text{or}$$

$$m = \frac{S.\phi'\lambda\phi'\mu\phi'\nu}{S.\lambda\mu\nu}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

[That this quantity is independent of the particular vectors λ, μ, ν is evident from the fact that if

$$\lambda' = p\lambda + q\mu + r\nu, \quad \mu' = p_1\lambda + q_1\mu + r_1\nu, \quad \text{and} \quad \nu' = p_2\lambda + q_2\mu + r_2\nu$$

be any other three vectors (which is possible since λ, μ, ν are not coplanar), we have

$$\phi'\lambda' = p\phi'\lambda + q\phi'\mu + r\phi'\nu, \text{ \&c., \&c.};$$

from which we deduce

$$S.\phi'\lambda'\phi'\mu'\phi'\nu' = \begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} S.\phi'\lambda\phi'\mu\phi'\nu,$$

and

$$S.\lambda'\mu'\nu' = \begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} S.\lambda\mu\nu,$$

so that the numerator and denominator of the fraction which expresses m are altered in the same ratio. Each of these quantities is in fact an *Invariant*, and the numerical multiplier is the same for both when we pass from any one set of three vectors to another.]

147. Let us now change ϕ to $\phi + g$, where g is any scalar. It is evident that ϕ' becomes $\phi' + g$, and our equation (4) becomes

$$\begin{aligned} m_g(\phi+g)^{-1}V\lambda\mu &= V(\phi'+g)\lambda(\phi'+g)\mu; \\ &= V\phi'\lambda\phi'\mu + gV(\phi'\lambda\mu+\lambda\phi'\mu)+g^2V\lambda\mu, \\ &= (m\phi^{-1}+g\chi+g^2)V\lambda\mu \text{ suppose.} \end{aligned}$$

In the above equation

$$\begin{aligned} m_g &= \frac{S.(\phi'+g)\lambda(\phi'+g)\mu(\phi'+g)\nu}{S.\lambda\mu\nu} \\ &= m+m_1g+m_2g^2+g^3 \end{aligned}$$

is what m becomes when ϕ is changed into $\phi+g$; m_1 and m_2 being two new scalar constants whose values are

$$m_1 = \frac{S.(\lambda\phi'\mu\phi'\nu+\phi'\lambda\mu\phi'\nu+\phi'\lambda\phi'\mu\nu)}{S.\lambda\mu\nu},$$

$$m_2 = \frac{S.(\lambda\mu\phi'\nu+\phi'\lambda\mu\nu+\lambda\phi'\mu\nu)}{S.\lambda\mu\nu}.$$

Substituting for m_g, and equating the coefficients of the various powers of g after operating on both sides by $\phi+g$, we have two identities and the following two equations,

$$\begin{aligned} m_2 &= \phi+\chi, \\ m_1 &= \phi\chi+m\phi^{-1}. \end{aligned}$$

[The first determines χ, and shows that we were justified in treating $V(\phi'\lambda\mu+\lambda\phi'\mu)$ as a linear and vector function of $V.\lambda\mu$. The result might have been also obtained thus,

$$\begin{aligned} S.\lambda\chi V\lambda\mu &= S.\lambda\phi'\lambda\mu = -S.\lambda\mu\phi'\lambda = -S.\lambda\phi V\lambda\mu, \\ S.\mu\chi V\lambda\mu &= S.\mu\lambda\phi'\mu = -S.\mu\phi V\lambda\mu, \\ S.\nu\chi V\lambda\mu &= S.(\nu\phi'\lambda\mu+\nu\lambda\phi'\mu) \\ &= m_2S\lambda\mu\nu - S.\lambda\mu\phi'\nu \\ &= S.\nu(m_2V\lambda\mu - \phi V\lambda\mu); \end{aligned}$$

and all three are satisfied by

$$\chi = m_2-\phi.]$$

148. Eliminating χ from these equations we find

$$\begin{aligned} m_1 &= \phi(m_2-\phi)+m\phi^{-1}, \\ \text{or} \quad m\phi^{-1} &= m_1-m_2\phi+\phi^2, \end{aligned}$$

which contains the complete solution of linear and vector equations.

149. More to satisfy the student of the validity of the above investigation, about whose logic he may at first feel some difficulties, than to obtain easy solutions, we take a few very simple examples to begin with: and we append for comparison easy solutions obtained by methods specially adapted to each case.

150. *Example I.*

Let $\phi\rho = V.\alpha\rho\beta = \gamma$.

Then $\phi'\rho = V.\beta\rho\alpha = \phi\rho$.

Hence $$m = \frac{1}{S.\lambda\mu\nu} S(V.\alpha\lambda\beta\, V.\alpha\mu\beta\, V.\alpha\nu\beta).$$

Now λ, μ, ν are any three non-coplanar vectors; and we may therefore put for them α, β, γ *if the latter be non-coplanar.*

With this proviso

$$\begin{aligned} m &= \frac{1}{S.\alpha\beta\gamma} S(\alpha^2\beta.\alpha\beta^2.\, V.\alpha\gamma\beta) \\ &= \alpha^2\beta^2 S\alpha\beta, \\ m_1 &= \frac{1}{S.\alpha\beta\gamma} S(\alpha^2\beta.\alpha\beta^2.\gamma + \alpha.\alpha\beta^2.\, V.\alpha\gamma\beta + \alpha^2\beta.\beta.\, V.\alpha\gamma\beta) \\ &= -\alpha^2\beta^2, \\ m_2 &= \frac{1}{S.\alpha\beta\gamma} S(\alpha^2\beta.\beta.\gamma + \alpha.\alpha\beta^2.\gamma + \alpha\beta\, V.\alpha\gamma\beta) \\ &= -S\alpha\beta. \end{aligned}$$

Hence

$$\alpha^2\beta^2 S\alpha\beta.\phi^{-1}\gamma = -\alpha^2\beta^2 S\alpha\beta.\rho = -\alpha^2\beta^2\gamma + S\alpha\beta\, V.\alpha\gamma\beta + V.\alpha(V.\alpha\gamma\beta)\beta,$$

which is one form of solution.

By expanding the vectors of products we may easily reduce it to the form

$$\alpha^2\beta^2 S\alpha\beta.\rho = -\alpha^2\beta^2\gamma + \alpha\beta^2 S\alpha\gamma + \beta\alpha^2 S\beta\gamma,$$

or $$\rho = \frac{\alpha^{-1}S\alpha\gamma + \beta^{-1}S\beta\gamma - \gamma}{S\alpha\beta}.$$

151. To verify this solution, we have

$$V.\alpha\rho\beta = \frac{1}{S\alpha\beta}(\beta S\alpha\gamma + \alpha S\beta\gamma - V.\alpha\gamma\beta) = \gamma,$$

which is the given equation.

152. An easier mode of arriving at the same solution, in this simple case, is as follows :—

Operating by $S.\alpha$ and $S.\beta$ on the given equation

$$V.\alpha\rho\beta = \gamma,$$

we obtain

$$\alpha^2 S\beta\rho = S\alpha\gamma,$$

$$\beta^2 S\alpha\rho = S\beta\gamma;$$

and therefore

$$\alpha S\beta\rho = \alpha^{-1}S\alpha\gamma,$$

$$\beta S\alpha\rho = \beta^{-1}S\beta\gamma.$$

But the given equation may be written

$$\alpha S\beta\rho - \rho S\alpha\beta + \beta S\alpha\rho = \gamma.$$

Substituting and transposing we get

$$\rho S\alpha\beta = \alpha^{-1}S\alpha\gamma + \beta^{-1}S\beta\gamma - \gamma,$$

which agrees with the result of § 150.

153. If α, β, γ be coplanar, the above mode of solution is applicable, but the result may be deduced much more simply.

For (§ 101) $S.\alpha\beta\gamma = 0$, and the equation then gives $S.\alpha\beta\rho = 0$, so that ρ is also coplanar with α, β, γ.

Hence the equation may be written

$$\alpha\rho\beta = \gamma,$$

and at once

$$\rho = \alpha^{-1}\gamma\beta^{-1};$$

and this, being a vector, may be written

$$= \alpha^{-1}S\beta^{-1}\gamma + \beta^{-1}S\alpha^{-1}\gamma - \gamma S\alpha^{-1}\beta^{-1}.$$

This formula is *equivalent* to that just given, but not equal to it term by term. [The student will find it a good exercise to prove *directly* that, if α, β, γ are coplanar, we have

$$\frac{1}{S\alpha\beta}(\alpha^{-1}S\alpha\gamma+\beta^{-1}S\beta\gamma-\gamma)=\alpha^{-1}S\beta^{-1}\gamma+\beta^{-1}S\alpha^{-1}\gamma-\gamma S\alpha^{-1}\beta^{-1}.]$$

154. *Example II.*

Let $\phi\rho = V.\alpha\beta\rho = \gamma.$

Suppose α, β, γ not to be coplanar, and employ them as λ, μ, ν to calculate the coefficients in the equation for ϕ^{-1}. We have

$$S.\sigma\phi\rho = S.\sigma\alpha\beta\rho = S.\rho V.\sigma\alpha\beta = S.\rho\phi'\sigma.$$

Hence $\phi'\rho = V.\rho\alpha\beta = V.\beta\alpha\rho.$

We have now

$$m = \frac{1}{S.\alpha\beta\gamma}S(\beta\alpha^2.\beta\alpha\beta.V.\beta\alpha\gamma) = \frac{\alpha^2\beta^2}{S.\alpha\beta\gamma}S.\alpha\beta V.\beta\alpha\gamma$$
$$= \alpha^2\beta^2 S\alpha\beta.$$

$$m_1 = \frac{1}{S.\alpha\beta\gamma}S(\alpha.\beta\alpha\beta.V.\beta\alpha\gamma+\beta\alpha^2.\beta.V.\beta\alpha\gamma+\beta\alpha^2.\beta\alpha\beta.\gamma)$$
$$= 2(S\alpha\beta)^2+\alpha^2\beta^2,$$

$$m_2 = \frac{1}{S.\alpha\beta\gamma}S(\alpha.\beta.V.\beta\alpha\gamma+\alpha.\beta\alpha\beta.\gamma+\beta\alpha^2.\beta.\gamma)$$
$$= 3S\alpha\beta.$$

Hence

$$\alpha^2\beta^2 S\alpha\beta.\phi^{-1}\gamma = \alpha^2\beta^2 S\alpha\beta.\rho$$
$$= (2(S\alpha\beta)^2+\alpha^2\beta^2)\gamma-3S\alpha\beta V.\alpha\beta\gamma+V.\alpha\beta V.\alpha\beta\gamma,$$

which, by expanding the vectors of products, takes easily the simpler form

$$\alpha^2\beta^2 S\alpha\beta.\rho = \alpha^2\beta^2\gamma-\alpha\beta^2 S\alpha\gamma+2\beta S\alpha\beta S\alpha\gamma-\beta\alpha^2 S\beta\gamma.$$

155. To verify this, operate by $V.\alpha\beta$ on both sides, and we have

$$\alpha^2\beta^2 S\alpha\beta\, V.\alpha\beta\rho = \alpha^2\beta^2 V.\alpha\beta\gamma - V.\alpha\beta\alpha\beta^2 S\alpha\gamma + 2\alpha\beta^2 S\alpha\beta S\alpha\gamma - \alpha\alpha^2\beta^2 S\beta\gamma$$

$$= \alpha^2\beta^2(\alpha S\beta\gamma - \beta S\alpha\gamma + \gamma S\alpha\beta) - (2\alpha S\alpha\beta - \beta\alpha^2)\beta^2 S\alpha\gamma$$

$$+ 2\alpha\beta^2 S\alpha\beta S\alpha\gamma - \alpha\alpha^2\beta^2 S\beta\gamma$$

$$= \alpha^2\beta^2 S\alpha\beta.\gamma,$$

or $$V.\alpha\beta\rho = \gamma.$$

156. To solve the same equation without employing the general method, we may proceed as follows:—

$$\gamma = V.\alpha\beta\rho = \rho S\alpha\beta + V.V(\alpha\beta)\rho.$$

Operating by $S.V\alpha\beta$ we have

$$S.\alpha\beta\gamma = S.\alpha\beta\rho S\alpha\beta.$$

Divide this by $S\alpha\beta$, and add it to the given equation. We thus obtain

$$\gamma + \frac{S.\alpha\beta\gamma}{S\alpha\beta} = \rho S\alpha\beta + V.V(\alpha\beta)\rho + S.V(\alpha\beta)\rho,$$

$$= (S\alpha\beta + V\alpha\beta)\rho,$$

$$= \alpha\beta\rho.$$

Hence $$\rho = \beta^{-1}\alpha^{-1}\left(\gamma + \frac{S.\alpha\beta\gamma}{S\alpha\beta}\right),$$

a form of solution somewhat simpler than that before obtained.

To show that they agree, however, let us multiply by $\alpha^2\beta^2 S\alpha\beta$, and we get

$$\alpha^2\beta^2 S\alpha\beta.\rho = \beta\alpha\gamma S\alpha\beta + \beta\alpha S.\alpha\beta\gamma.$$

In this form we see at once that the right-hand side is a vector, since its scalar is evidently zero (§ 89). Hence we may write

$$\alpha^2\beta^2 S\alpha\beta.\rho = V.\beta\alpha\gamma S\alpha\beta - V\alpha\beta S.\alpha\beta\gamma.$$

But by (3) of § 91,

$$-\gamma S.\alpha\beta V\alpha\beta + \alpha S.\beta(V\alpha\beta)\gamma + \beta S.V(\alpha\beta)\alpha\gamma + V\alpha\beta S.\alpha\beta\gamma = 0.$$

Add this to the right-hand side, and we have

$$\alpha^2\beta^2 S\alpha\beta.\rho = \gamma\left((S\alpha\beta)^2 - S.\alpha\beta V\alpha\beta\right) - \alpha\left(S\alpha\beta S\beta\gamma - S.\beta(V\alpha\beta)\gamma\right)$$

$$+ \beta\left(S\alpha\beta S\alpha\gamma + S.V(\alpha\beta)\alpha\gamma\right).$$

But $\quad (S\alpha\beta)^2 - S.\alpha\beta V\alpha\beta = (S\alpha\beta)^2 - (V\alpha\beta)^2 = \alpha^2\beta^2,$

$$S\alpha\beta S\beta\gamma - S.\beta(V\alpha\beta)\gamma = S\alpha\beta S\beta\gamma - S\beta\alpha S\beta\gamma + \beta^2 S\alpha\gamma = \beta^2 S\alpha\gamma$$

$$S\alpha\beta S\alpha\gamma + S.V(\alpha\beta)\alpha\gamma = S\alpha\beta S\alpha\gamma + S\alpha\beta S\alpha\gamma - \alpha^2 S\beta\gamma$$
$$= 2 S\alpha\beta S\alpha\gamma - \alpha^2 S\beta\gamma;$$

and the substitution of these values renders our equation identical with that of § 154.

[If α, β, γ be coplanar, the simplified forms of the expression for ρ lead to the equation

$$S\alpha\beta.\beta^{-1}\alpha^{-1}\gamma = \gamma - \alpha^{-1}S\alpha\gamma + 2\beta S\alpha^{-1}\beta^{-1}S\alpha\gamma - \beta^{-1}S\beta\gamma,$$

which, as before, we leave as an exercise to the student.]

157. *Example III.* The solution of the equation

$$V\epsilon\rho = \gamma$$

leads to the vanishing of some of the quantities m. Before, however, treating it by the general method, we shall deduce its solution from that of

$$V.\alpha\beta\rho = \gamma$$

already given. Our reason for so doing is that we thus have an opportunity of showing the nature of some of the cases in which one or more of m, m_1, m_2 vanish; and also of introducing an example of the use of vanishing fractions in quaternions. Far simpler solutions will be given in the following sections.

The solution of the last-written equation is, § 154,

$$\alpha^2\beta^2 S\alpha\beta.\rho = \alpha^2\beta^2\gamma - \alpha\beta^2 S\alpha\gamma - \beta\alpha^2 S\beta\gamma + 2\beta S\alpha\beta S\alpha\gamma.$$

If we now put

$$\alpha\beta = e + \epsilon$$

where e is a scalar, the solution of the first-written equation will evidently be derived from that of the second by making e gradually tend to zero.

We have, for this purpose, the following necessary transformations :—

$$\alpha^2\beta^2 = \alpha\beta K.\alpha\beta = (e+\epsilon)(e-\epsilon) = e^2 - \epsilon^2,$$

$$\begin{aligned}\alpha\beta^2 S\alpha\gamma + \beta\alpha^2 S\beta\gamma &= \alpha\beta.\beta S\alpha\gamma + \beta\alpha.\alpha S\beta\gamma,\\ &= (e+\epsilon)\beta S\alpha\gamma + (e-\epsilon)\alpha S\beta\gamma,\\ &= e(\beta S\alpha\gamma + \alpha S\beta\gamma) + \epsilon V.\gamma V\alpha\beta,\\ &= e(\beta S\alpha\gamma + \alpha S\beta\gamma) + \epsilon V\gamma\epsilon.\end{aligned}$$

Hence the solution becomes

$$\begin{aligned}(e^2-\epsilon^2)e\rho &= (e^2-\epsilon^2)\gamma - e(\beta S\alpha\gamma + \alpha S\beta\gamma) - \epsilon V\gamma\epsilon + 2e\beta S\alpha\gamma,\\ &= (e^2-\epsilon^2)\gamma + eV.\gamma V\alpha\beta - \epsilon V\gamma\epsilon,\\ &= (e^2-\epsilon^2)\gamma + eV\gamma\epsilon + \gamma\epsilon^2 - \epsilon S\gamma\epsilon,\\ &= e^2\gamma + eV\gamma\epsilon - \epsilon S\gamma\epsilon.\end{aligned}$$

Dividing by e, and then putting $e = 0$, we have

$$-\epsilon^2\rho = V\gamma\epsilon - \epsilon\mathcal{L}_0\left(\frac{S\gamma\epsilon}{e}\right).$$

Now, by the form of the given equation, we see that

$$S\gamma\epsilon = 0.$$

Hence the limit is indeterminate, and we may put for it x, where x is *any* scalar. Our solution is, therefore,

$$\rho = -V\frac{\gamma}{\epsilon} + x\epsilon^{-1};$$

or, as it may be written, since $S\gamma\epsilon = 0$,

$$\rho = \epsilon^{-1}(\gamma + x).$$

The verification is obvious—for we have

$$\epsilon\rho = \gamma + x.$$

158. This suggests a very simple mode of solution. For we see that

$$V\epsilon\rho = V\epsilon(\rho - x\epsilon^{-1}) = \gamma,$$

a constant *vector* whatever x may be. But the vector sign may now be *removed* as unnecessary, so that we have

$$\epsilon(\rho - x\epsilon^{-1}) = \gamma,$$

or $$\rho = \epsilon^{-1}(\gamma + x),$$

if, and only if, ρ satisfies the equation

$$V\epsilon\rho = \gamma.$$

159. To apply the general method, we may take ϵ, γ and $\epsilon\gamma$ (which is a vector) for λ, μ, ν.

We find $\phi'\rho = V\rho\epsilon$.

Hence
$$m = 0,$$
$$m_1 = -\frac{1}{\epsilon^2\gamma^2} S.(\epsilon.\epsilon\gamma.\epsilon^2\gamma) = -\epsilon^2,$$
$$m_2 = 0.$$

Hence
$$-\epsilon^2\phi + \phi^3 = 0,$$
or
$$\phi^{-1} = \frac{1}{\epsilon^2}\phi + \phi^{-2}0.$$

That is,
$$\rho = \frac{1}{\epsilon^2} V\epsilon\gamma + x\epsilon,$$
$$= \epsilon^{-1}\gamma + x\epsilon, \text{ as before.}$$

Our warrant for putting $x\epsilon$ as the equivalent of $\phi^{-2}0$ is this:—

The equation
$$\phi^2\sigma = 0$$
may be written
$$V.\epsilon V\epsilon\sigma = 0 = \sigma\epsilon^2 - \epsilon S\epsilon\sigma.$$

Hence, unless $\sigma = 0$, we have $\sigma \parallel \epsilon = x\epsilon$.

160. *Example IV.* As a final example let us take the most general form of ϕ, which, as will be soon proved, may be expressed as follows:—
$$\phi\rho = \alpha S\beta\rho + \alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho = \gamma.$$

Here
$$\phi'\rho = \beta S\alpha\rho + \beta_1 S\alpha_1\rho + \beta_2 S\alpha_2\rho,$$
and, consequently, taking α, α_1, α_2, which are in this case non-coplanar vectors, for λ, μ, ν, we have
$$m = \frac{1}{S.\alpha\alpha_1\alpha_2} S.(\beta S\alpha\alpha + \beta_1 S\alpha_1\alpha + \beta_2 S\alpha_2\alpha)(\beta S\alpha\alpha_1 + \beta_1 S\alpha_1\alpha_1 + \ldots)(\beta S\alpha\alpha_2 + \ldots\ldots)$$
$$= \frac{S.\beta\beta_1\beta_2}{S.\alpha\alpha_1\alpha_2} \begin{vmatrix} S\alpha\alpha & S\alpha_1\alpha & S\alpha_2\alpha \\ S\alpha\alpha_1 & S\alpha_1\alpha_1 & S\alpha_2\alpha_1 \\ S\alpha\alpha_2 & S\alpha_1\alpha_2 & S\alpha_2\alpha_2 \end{vmatrix}$$
$$= \frac{S.\beta\beta_1\beta_2}{S.\alpha\alpha_1\alpha_2} (AS\alpha\alpha + A_1 S\alpha_1\alpha + A_2 S\alpha_2\alpha),$$

where
$$\begin{aligned}A &= S\alpha_1\alpha_1 S\alpha_2\alpha_2 - S\alpha_2\alpha_1 S\alpha_1\alpha_2\\ &= -S.V\alpha_1\alpha_2 V\alpha_1\alpha_2\\ A_1 &= S\alpha_2\alpha_1\, S\alpha\alpha_2 - S\alpha\alpha_1 S\alpha_2\alpha_2\\ &= -S.V\alpha_2\alpha V\alpha_1\alpha_2\\ A_2 &= S\alpha\alpha_1\, S\alpha_1\alpha_2 - S\alpha_1\alpha_1 S\alpha\alpha_2\\ &= -S.V\alpha\alpha_1\, V\alpha_1\alpha_2.\end{aligned}$$

Hence the value of the determinant is

$$-(S\alpha\alpha S.V\alpha_1\alpha_2\, V\alpha_1\alpha_2 + S\alpha_1\alpha S.V\alpha_2\alpha\, V\alpha_1\alpha_2 + S\alpha_2\alpha S.V\alpha\alpha_1\, V\alpha_1\alpha_2)$$
$$= -S.\alpha(V\alpha_1\alpha_2 S.\alpha\alpha_1\alpha_2)\ \{\text{by § 92 (4)}\} = -(S.\alpha\alpha_1\alpha_2)^2.$$

The interpretation of this result in spherical trigonometry is very interesting.

By it we see that

$$m = -S.\alpha\alpha_1\alpha_2 S.\beta\beta_1\beta_2.$$

Similarly,

$$m_1 = \frac{1}{S.\alpha\alpha_1\alpha_2} S.[\alpha(\beta S\alpha\alpha_1 + \beta_1 S\alpha_1\alpha_1 + \beta_2 S\alpha_2\alpha_1)(\beta S\alpha\alpha_2 + \beta_1 S\alpha_1\alpha_2 + \beta_2 S\alpha_2\alpha_2) + \&\text{c.}]$$

$$= \frac{1}{S.\alpha\alpha_1\alpha_2}(S.\alpha\beta\beta_1(S\alpha\alpha_1 S\alpha_1\alpha_2 - S\alpha_1\alpha_1 S\alpha\alpha_2) + \ldots\ldots)$$

$$= \frac{1}{S.\alpha\alpha_1\alpha_2}(S.\alpha\beta\beta_1 S.\alpha V.\alpha_1 V\alpha_2\alpha_1 + \ldots\ldots)$$

$$\begin{aligned}= -\frac{1}{S.\alpha\alpha_1\alpha_2}[S.\alpha(&V\beta\beta_1 S.V\alpha\alpha_1 V\alpha_1\alpha_2 + V\beta_2\beta S.V\alpha_2\alpha V\alpha_1\alpha_2 + V\beta_1\beta_2 S.V\alpha_1\alpha_2 V\alpha_1\alpha_2)\\ &+ S.\alpha_1(V\beta\beta_1 S.V\alpha\alpha_1 V\alpha_2\alpha + \ldots\ldots)\\ &+ S.\alpha_2(V\beta\beta_1 S.V\alpha\alpha_1 V\alpha\alpha_1 + \ldots\ldots)];\end{aligned}$$

or, taking the terms by columns instead of by rows,

$$\begin{aligned}= -\frac{1}{S.\alpha\alpha_1\alpha_2}[S.V\beta\beta_1(&\alpha S.V\alpha\alpha_1 V\alpha_1\alpha_2 + \alpha_1 S.V\alpha\alpha_1 V\alpha_2\alpha + \alpha_2 S.V\alpha\alpha_1 V\alpha\alpha_1)\\ &+ \quad .\ .\ .\ .\ .\ .\ . \quad + \quad .\ .\ .\ .\ .\ .\ ],\end{aligned}$$

$$= -\frac{1}{S.\alpha\alpha_1\alpha_2}[S.V\beta\beta_1(V\alpha\alpha_1 S.\alpha\alpha_1\alpha_2) \quad + \quad .\ .\ .\ .\ .\ .\ ],$$

$$= -S(V\alpha\alpha_1 V\beta\beta_1 + V\alpha_1\alpha_2 V\beta_1\beta_2 + V\alpha_2\alpha_1 V\beta_2\beta_1).$$

Again,

$$m_2 = \frac{1}{S.\alpha\alpha_1\alpha_2} S\left[\alpha\alpha_1(\beta S\alpha\alpha_2 + \beta_1 S\alpha_1\alpha_2 + \ldots) + \alpha_2\alpha(\beta S\alpha\alpha_1 + \ldots) + \alpha_1\alpha_2(\beta S\alpha\alpha + \ldots)\right],$$

or, grouping as before,

$$= -\frac{1}{S.\alpha\alpha_1\alpha_2} S\left[\beta(V\alpha\alpha_1 S\alpha\alpha_2 + V\alpha_2\alpha S\alpha\alpha_1 + V\alpha_1\alpha_2 S\alpha\alpha) + \ldots\right],$$

$$= -\frac{1}{S.\alpha\alpha_1\alpha_2} S\left[\beta(\alpha S.\alpha\alpha_1\alpha_2) + \ldots\ldots\right] \quad (\S\ 92\ (4)),$$

$$= -S(\alpha\beta + \alpha_1\beta_1 + \alpha_2\beta_2).$$

And the solution is, therefore,

$$-S.\alpha\alpha_1\alpha_2 S.\beta\beta_1\beta_2.\phi^{-1}\gamma = -S.\alpha\alpha_1\alpha_2 S.\beta\beta_1\beta_2.\rho$$
$$= -\gamma\Sigma S.V\alpha\alpha_1 V\beta\beta_1 + \Sigma S\alpha\beta.\phi\gamma + \phi^2\gamma.$$

[It will be excellent practice for the student to work out in detail the blank portions of the above investigation, and also to prove directly that the value of ρ we have just found satisfies the given equation.]

161. But it is not necessary to go through such a long process to get the solution—though it will be advantageous to the student to read it carefully—for if we operate on the equation by $S.\alpha_1\alpha_2$, $S.\alpha_2\alpha$, and $S.\alpha\alpha_1$ we get

$$S.\alpha_1\alpha_2\alpha S\beta\rho = S.\alpha_1\alpha_2\gamma,$$
$$S.\alpha_2\alpha\alpha_1 S\beta_1\rho = S.\alpha_2\alpha\gamma,$$
$$S.\alpha\alpha_1\alpha_2 S\beta_2\rho = S.\alpha\alpha_1\gamma.$$

From these, by § 92 (4), we have at once

$$S.\alpha\alpha_1\alpha_2 S.\beta\beta_1\beta_2.\rho = V\beta\beta_1 S.\alpha\alpha_1\gamma + V\beta_1\beta_2 S.\alpha_1\alpha_2\gamma + V\beta_2\beta S.\alpha_2\alpha\gamma.$$

The student will find it a useful exercise to prove that this is equivalent to the solution in § 160.

To verify the present solution we have

$$S.\alpha\alpha_1\alpha_2 S.\beta\beta_1\beta_2(\alpha S\beta\rho + \alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho)$$
$$= \alpha S.\beta\beta_1\beta_2 S.\alpha_1\alpha_2\gamma + \alpha_1 S.\beta_1\beta_2\beta S.\alpha_2\alpha\gamma + \alpha_2 S.\beta_2\beta\beta_1 S.\alpha\alpha_1\gamma$$
$$= S.\beta\beta_1\beta_2(\gamma S.\alpha\alpha_1\alpha_2), \text{ by § 91 (3).}$$

162. It is evident, from these examples, that for special cases we can usually find modes of solution of the linear and vector equation which are simpler in application than the general process of § 148. The real value of that process however consists partly in its enabling us to express inverse functions of ϕ, such as $(\phi+g)^{-1}$ for instance, in terms of direct operations, a property which will be of great use to us later; partly in its leading us to the fundamental cubic

$$\phi^3-m_2\phi^2+m_1\phi-m=0,$$

which is an immediate deduction from the equation of § 148, and whose interpretation is of the utmost importance with reference to the axes of surfaces of the second order, principal axes of inertia, the analysis of strains in a distorted solid, and various similar enquiries.

163. When the function ϕ *is its own conjugate*, that is, when

$$S\rho\phi\sigma=S\sigma\phi\rho$$

for all values of ρ and σ, the vectors for which

$$(\phi-g)\rho=0$$

form in general a real and definite rectangular system. This, of course, may in particular cases degrade into one definite vector, and *any* pair of others perpendicular to it; and cases may occur in which the equation is satisfied for *every* vector.

Suppose the roots of $m_g=0$ (§ 147) to be real and different, then

$$\left.\begin{aligned}\phi\rho_1&=g_1\rho_1\\ \phi\rho_2&=g_2\rho_2\\ \phi\rho_3&=g_3\rho_3\end{aligned}\right\}\text{ where }\rho_1,\ \rho_2,\ \rho_3\text{ are definite vectors.}$$

Hence $$g_1g_2S\rho_1\rho_2=S.\phi\rho_1\phi\rho_2$$
$$=S.\rho_1\phi^2\rho_2,\text{ or }=S.\rho_2\phi^2\rho_1,$$

because ϕ is its own conjugate.

But $$\phi^2\rho_2=g_2^2\rho_2,$$
$$\phi^2\rho_1=g_1^2\rho_1,$$

and therefore

$$g_1 g_2 S\rho_1\rho_2 = g_2^2 S\rho_1\rho_2 = g_1^2 S\rho_1\rho_2\,;$$

which, as g_1 and g_2 are by hypothesis different, requires

$$S\rho_1\rho_2 = 0.$$

Similarly $\qquad S\rho_2\rho_3 = 0, \qquad S\rho_3\rho_1 = 0.$

If two roots be equal, as g_2, g_3, we still have, by the above proof, $S\rho_1\rho_2 = 0$ and $S\rho_1\rho_3 = 0$. But there is nothing farther to determine ρ_2 and ρ_3, which are therefore *any* vectors perpendicular to ρ_1.

If all three roots be equal, *every* real vector satisfies the equation $(\phi - g)\rho = 0$.

164. Next, as to the *reality* of the three directions in this case.

Suppose $g_2 + h_2\sqrt{-1}$ to be a root, and let $\rho_2 + \sigma_2\sqrt{-1}$ be the corresponding value of ρ, where g_2 and h_2 are real numbers, ρ_2 and σ_2 real vectors, and $\sqrt{-1}$ the old imaginary of algebra.

Then $\qquad \phi(\rho_2 + \sigma_2\sqrt{-1}) = (g_2 + h_2\sqrt{-1})(\rho_2 + \sigma_2\sqrt{-1}),$

and this divides itself, as in algebra, into the two equations

$$\phi\rho_2 = g_2\rho_2 - h_2\sigma_2,$$
$$\phi\sigma_2 = h_2\rho_2 + g_2\sigma_2.$$

Operating on these by $S.\sigma_2$, $S.\rho_2$ respectively, and subtracting the results, remembering our condition as to the nature of ϕ

$$S\sigma_2\phi\rho_2 = S\rho_2\phi\sigma_2,$$

we have $\qquad h_2(\sigma_2^2 + \rho_2^2) = 0.$

But, as σ_2 and ρ_2 are both real vectors, the sum of their squares cannot vanish. Hence h_2 vanishes, and with it the impossible part of the root.

165. When ϕ is self-conjugate, we have shewn that the equation

$$g^3 - m_2 g^2 + m_1 g - m = 0$$

has three real roots, in general different from one another.

Hence the cubic in ϕ may be written

$$(\phi-g_1)(\phi-g_2)(\phi-g_3) = 0,$$

and in this form we can easily see the meaning of the cubic. For, let ρ_1, ρ_2, ρ_3 be such that

$$(\phi-g_1)\rho_1 = 0, \quad (\phi-g_2)\rho_2 = 0, \quad (\phi-g_3)\rho_3 = 0.$$

Then since any vector may be expressed by the equation

$$\rho S.\rho_1\rho_2\rho_3 = \rho_1 S.\rho_2\rho_3\rho + \rho_2 S.\rho_3\rho_1\rho + \rho_3 S.\rho_1\rho_2\rho \quad (\S\ 91),$$

we see that when the complex operation, denoted by the left-hand member of the above symbolic equation, is performed on ρ, the first of the three factors makes the term in ρ_1 vanish, the second and third those in ρ_2 and ρ_3 respectively. In other words, by the successive performance upon a vector of the operations $\phi-g_1$, $\phi-g_2$, $\phi-g_3$, it is deprived successively of its resolved parts in the directions of ρ_1, ρ_2, ρ_3 respectively; and is thus necessarily reduced to zero, since ρ_1, ρ_2, ρ_3 are (because we have supposed g_1, g_2, g_3 to be distinct) rectangular vectors.

166. If we take ρ_1, ρ_2, ρ_3 as rectangular *unit*-vectors, we have

$$-\rho = \rho_1 S\rho_1\rho + \rho_2 S\rho_2\rho + \rho_3 S\rho_3\rho,$$

whence $$\phi\rho = -g_1\rho_1 S\rho_1\rho - g_2\rho_2 S\rho_2\rho - g_3\rho_3 S\rho_3\rho;$$

or, still more simply, putting i, j, k for ρ_1, ρ_2, ρ_3, we find that *any* self-conjugate function may be thus expressed

$$\phi\rho = -g_1 iSi\rho - g_2 jSj\rho - g_3 kSk\rho,$$

provided, of course, i, j, k be taken as roots of the equation

$$V\rho\phi\rho = 0.$$

167. A very important transformation of the self-conjugate linear and vector function is easily derived from this form.

We have seen that it involves *three* scalar constants only, viz. g_1, g_2, g_3. Let us enquire, then, whether it can be reduced to the following form

$$\phi\rho = p\rho + qV.(i+ek)\rho(i-ek),$$

which also involves but three scalar constants p, q, e.

Substituting for ρ the equivalent

$$\rho = -iSi\rho - jSj\rho - kSk\rho,$$

expanding, and equating coefficients of i, j, k in the two expressions for $\phi\rho$, we find

$$-g_1 = -p + q(2-1+e^2),$$
$$-g_2 = -p - q(1-e^2),$$
$$-g_3 = -p - q(2e^2+1-e^2).$$

These give at once

$$-(g_1-g_2) = 2q,$$
$$-(g_2-g_3) = 2e^2q.$$

Hence, as we suppose the transformation to be real, and therefore e^2 to be positive, it is evident that g_1-g_2 and g_2-g_3 have the same sign; so that we must choose as auxiliary vectors in the last term of $\phi\rho$ those two of the rectangular directions i, j, k for which the coefficients g have the greatest and least values.

We have then

$$e^2 = \frac{g_2-g_3}{g_1-g_2},$$
$$q = -\tfrac{1}{2}(g_1-g_2),$$
and $$p = \tfrac{1}{2}(g_1+g_3).$$

168. We may, therefore, always determine definitely the vectors λ, μ, and the scalar p, in the equation

$$\phi\rho = p\rho + V.\lambda\rho\mu$$

when ϕ is self-conjugate, and the corresponding cubic has not equal roots, subject to the single restriction that

$$T.\lambda\mu$$

is known, but not the separate tensors of λ and μ. This result is important in the theory of surfaces of the second order, and will be developed in Chapter VII.

169. Another important transformation of ϕ when self-conjugate is the following,

$$\phi\rho = a\alpha V\alpha\rho + b\beta S\beta\rho,$$

where a and b are scalars, and α and β are unit-vectors. This, of course, involves six scalar constants, and belongs to the most general form

$$\phi\rho = -g_1\rho_1 S\rho_1\rho - g_2\rho_2 S\rho_2\rho - g_3\rho_3 S\rho_3\rho,$$

where ρ_1, ρ_2, ρ_3 are the rectangular unit-vectors for which ρ and $\phi\rho$ are parallel. We merely mention this form in passing, as it belongs to the *focal* transformation of the equation of surfaces of the second order, which will not be farther alluded to in this work. It will be a good exercise for the student to determine α, β, a and b, in terms of g_1, g_2, g_3, and ρ_1, ρ_2, ρ_3.

170. We cannot afford space for a detailed account of the singular properties of these vector functions, and will content ourselves with the enunciation and proof of one or two of the most important.

In the equation

$$m\phi^{-1}V\lambda\mu = V\phi'\lambda\phi'\mu \quad (\S\ 145),$$

substitute λ for $\phi'\lambda$ and μ for $\phi'\mu$, and we have

$$mV\phi'^{-1}\lambda\phi'^{-1}\mu = \phi V\lambda\mu.$$

Change ϕ to $\phi+g$, and therefore ϕ' to $\phi'+g$, and m to m_g, we have

$$m_g V(\phi'+g)^{-1}\lambda(\phi'+g)^{-1}\mu = (\phi+g)V\lambda\mu;$$

a formula which will be found to be of considerable use.

171. Again, by § 147,

$$\frac{m_g}{g}S.\rho(\phi+g)^{-1}\rho = \frac{m}{g}S\rho\phi^{-1}\rho + S\rho\chi\rho + g\rho^2.$$

Similarly

$$\frac{m_h}{h}S.\rho(\phi+h)^{-1}\rho = \frac{m}{h}S\rho\phi^{-1}\rho + S\rho\chi\rho + h\rho^2.$$

Hence

$$\frac{m_g}{g}S.\rho(\phi+g)^{-1}\rho - \frac{m_h}{h}S.\rho(\phi+h)^{-1}\rho = (g-h)\left\{\rho^2 - \frac{mS\rho\phi^{-1}\rho}{gh}\right\}.$$

That is, the functions

$$\frac{m_g}{g} S.\rho(\phi+g)^{-1}\rho, \quad \text{and} \quad \frac{m_h}{h} S.\rho(\phi+h)^{-1}\rho$$

are identical, i. e. *when equated to constants represent the same series of surfaces*, not merely when

$$g = h,$$

but also, whatever be g and h, if they be scalar functions of ρ which satisfy the equation

$$mS.\rho\phi^{-1}\rho = gh\rho^2.$$

This is a generalization, due to Hamilton, of a singular result obtained by the author *.

172. The equations

$$\left.\begin{array}{l} S.\rho(\phi+g)^{-1}\rho = 0, \\ S.\rho(\phi+h)^{-1}\rho = 0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

are equivalent to

$$mS\rho\phi^{-1}\rho + gS\rho\chi\rho + g^2\rho^2 = 0,$$

$$mS\rho\phi^{-1}\rho + hS\rho\chi\rho + h^2\rho^2 = 0.$$

Hence $\quad m(1-x)S\rho\phi^{-1}\rho + (g-hx)S\rho\chi\rho + (g^2-h^2x)\rho^2 = 0,$

whatever scalar be represented by x.

That is, the two equations (1) represent the *same* surface if this identity be satisfied. As particular cases let

(1) $x = 1$, in which case

$$S\rho^{-1}\chi\rho + g + h = 0.$$

(2) $g - hx = 0$, in which case

$$m\left(1-\frac{g}{h}\right)S\rho^{-1}\phi^{-1}\rho + \left(g^2 - h^2\frac{g}{h}\right) = 0,$$

or $\quad mS\rho^{-1}\phi^{-1}\rho - gh = 0.$

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* Note on the Cartesian equation of the Wave-Surface. *Quarterly Mathem. Journal*, Oct. 1859.

(3) $x = \frac{g^2}{h^2}$, giving

$$m\left(1-\frac{g^2}{h^2}\right)S\rho\phi^{-1}\rho + \left(g - h\frac{g^2}{h^2}\right)S\rho\chi\rho = 0,$$

or $$m(h+g)S\rho\phi^{-1}\rho + ghS\rho\chi\rho = 0.$$

173. In various investigations we meet with the quaternion

$$q = \alpha\phi\alpha + \beta\phi\beta + \gamma\phi\gamma,$$

where α, β, γ are three unit-vectors at right angles to each other. It admits of being put in a very simple form, which is occasionally of considerable importance.

We have, obviously, by the properties of a rectangular unit-system

$$q = \beta\gamma\phi\alpha + \gamma\alpha\phi\beta + \alpha\beta\phi\gamma.$$

As we have also

$$S.\alpha\beta\gamma = -1 \quad (\S\ 71\ (13)),$$

a glance at the formulæ of § 147 shows that

$$Sq = -m_2,$$

at least if ϕ be self-conjugate. If it be not, then (as will be shown in § 174)

$$\phi\rho = \phi'\rho + V\epsilon\rho,$$

and the new term disappears in Sq.

We have also, by § 90 (2),

$$\begin{aligned} Vq &= \alpha(S\beta\phi\gamma - S\gamma\phi\beta) + \beta(S\gamma\phi\alpha - S\alpha\phi\gamma) + \gamma(S\alpha\phi\beta - S\beta\phi\alpha) \\ &= \alpha S\beta(\phi-\phi')\gamma + \beta S\gamma(\phi-\phi')\alpha + \gamma S\alpha(\phi-\phi')\beta \\ &= \alpha S.\beta\epsilon\gamma + \beta S.\gamma\epsilon\alpha + \gamma S.\alpha\epsilon\beta \\ &= -(\alpha S\alpha\epsilon + \beta S\beta\epsilon + \gamma S\gamma\epsilon) = \epsilon. \end{aligned}$$

Many similar singular properties of ϕ in connection with a rectangular system might easily be given; for instance,

$$V(\alpha V\phi\beta\phi\gamma + \beta V\phi\gamma\phi\alpha + \gamma V\phi\alpha\phi\beta) = \phi\epsilon;$$

which the reader may easily verify by a process similar to that just given, or (more directly) by the help of § 145 (4). A few others will be found among the Examples appended to this Chapter.

174. To conclude, we may remark that as in many of the immediately preceding investigations we have supposed ϕ to be self-conjugate, a very simple step enables us to pass from this to the non-conjugate form.

For, if ϕ' be conjugate to ϕ, we have

$$S\rho\phi'\sigma = S\sigma\phi\rho,$$

and also

$$S\rho\phi\sigma = S\sigma\phi'\rho.$$

Adding, we have

$$S\rho(\phi+\phi')\sigma = S\sigma(\phi+\phi')\rho\,;$$

so that the function $(\phi+\phi')$ is self-conjugate.

Again, $$S\rho\phi\rho = S\rho\phi'\rho,$$

which gives $$S\rho(\phi-\phi')\rho = 0.$$

Hence $$(\phi-\phi')\rho = V\epsilon\rho,$$

where ϵ is some real vector, and therefore

$$\begin{aligned}\phi\rho &= \tfrac{1}{2}(\phi+\phi')\rho+\tfrac{1}{2}(\phi-\phi')\rho\\ &= \tfrac{1}{2}(\phi+\phi')\rho+\tfrac{1}{2}V\epsilon\rho.\end{aligned}$$

Thus *every non-conjugate linear and vector function differs from a conjugate function solely by a term of the form*

$$V\epsilon\rho.$$

The geometric signification of this will be found in the Chapter on Kinematics.

175. We have shown, at some length, how a linear and vector equation containing an unknown *vector* is to be solved in the most general case; and this, by § 138, shows how to find an unknown *quaternion* from any sufficiently general linear equation containing it. That such an equation may be sufficiently general it must have both scalar and vector parts: the first gives *one,* and the second *three,* scalar equations; and these are required to determine completely the four scalar elements of the unknown quaternion.

176. Thus

$$Tq = a$$

being but one scalar equation, gives

$$q = aUr,$$

where r is any quaternion whatever.

Similarly

$$Sq = a$$

gives

$$q = a + \theta,$$

where θ is any vector whatever. In each of these cases, only one scalar condition being given, the solution contains three scalar indeterminates.

177. Again, the reader may easily prove that

$$V.\alpha Vq = \beta,$$

where α is a given vector, gives

$$V\alpha q = \beta + x\alpha.$$

Hence, assuming

$$S\alpha q = y,$$

we have

$$\alpha q = y + x\alpha + \beta,$$

or

$$q = x + y\alpha^{-1} + \alpha^{-1}\beta.$$

Here, the given equation being equivalent to two scalar conditions, the solution contains two scalar indeterminates.

178. Next take the equation

$$V\alpha q = \beta.$$

Operating by $S.\alpha^{-1}$, we get

$$Sq = S\alpha^{-1}\beta,$$

so that the given equation becomes

$$V\alpha(S\alpha^{-1}\beta + Vq) = \beta,$$

or

$$V\alpha Vq = \beta - \alpha S\alpha^{-1}\beta = \alpha V\alpha^{-1}\beta.$$

From this, by § 158, we see that

$$Vq = a^{-1}(x + aVa^{-1}\beta),$$

whence

$$q = Sa^{-1}\beta + a^{-1}(x + aVa^{-1}\beta)$$
$$= a^{-1}(\beta + x),$$

and, the given equation being equivalent to three scalar conditions, but one undetermined scalar remains in the value of q.

This solution might have been obtained at once, since our equation gives merely the *vector* of the quaternion aq, and leaves its scalar undetermined.

Hence, taking x for the scalar, we have

$$aq = Saq + Vaq$$
$$= x + \beta.$$

179. Finally, of course, from

$$aq = \beta,$$

which is equivalent to four scalar equations, we obtain a definite value of the unknown quaternion in the form

$$q = a^{-1}\beta.$$

180. Before taking leave of linear equations, we may mention that Hamilton has shown how to solve any linear equation containing an unknown *quaternion*, by a process analogous to that which he employed to determine an unknown *vector* from a linear and vector equation; and to which a large part of this Chapter has been devoted. Besides the increased complexity, the peculiar feature disclosed by this beautiful discovery is that the symbolic equation for a linear quaternion function, corresponding to the cubic in ϕ of § 162, is a *biquadratic*, so that the inverse function is given in terms of the first, second, and third powers of the direct function. In an elementary work like the present the discussion of such a question would be out of place: although it is not very difficult to derive the more general result by an application of processes already explained. The reader is therefore referred to the *Elements of Quaternions*, p. 491.

181. The solution of the following frequently-occurring particular form of linear quaternion equation

$$aq+qb = c,$$

where a, b, and c are any given quaternions, has been effected by Hamilton by an ingenious process, which was applied in § 133 (5) to a simple case.

Multiply the whole *by* Ka, and *into* b, and we have

$$T^2a.q+Ka.qb = Ka.c,$$

$$\text{and} \qquad aqb+qb^2 = cb.$$

Adding, we have

$$q(T^2a+b^2+2Sa.b) = Ka.c+cb,$$

from which q is at once found.

To this form any equation such as

$$a'qb'+c'qd'=e'$$

can of course be reduced, by multiplication *by* $(c')^{-1}$ and *into* $(b')^{-1}$.

182. As another example, let us find the differential of the cube root of a quaternion. If

$$q^3 = r$$

we have $$q^2dq+qdq.q+dq.q^2 = dr.$$

Multiply *by* q, and *into* q^{-1}, simultaneously, and we obtain

$$q^3dq.q^{-1}+q^2dq+qdq.q = qdr.q^{-1}.$$

Subtracting this from the preceding equation we have

$$dq.q^2-q^3dq.q^{-1} = dr-qdr.q^{-1},$$

from which dq, or $d(r^{\frac{1}{3}})$, can be found by the process of last section.

The method here employed can be easily applied to find the differential of any root of a quaternion.

183. To show some of the characteristic peculiarities in the solution even of quaternion equations of the first degree when they are not sufficiently general, let us take the very simple one

$$aq = qb,$$

and give every step of the solution, as practice in transformations.

Apply Hamilton's process (§ 181), and we get

$$T^2a.q = Ka.qb,$$

$$qb^2 = aqb.$$

These give $$q(T^2a+b^2-2bSa) = 0,$$

so that the equation gives no real finite value for q unless

$$T^2a+b^2-2bSa = 0,$$

or $$b = Sa+\beta TVa$$

where β is some unit-vector.

By a similar process we may evidently show that

$$a = Sb+\alpha TVb,$$

α being another unit-vector.

But, by the given equation,

$$Ta = Tb,$$

or $$S^2a+T^2Va = S^2b+T^2Vb;$$

from which, and the above values of a and b, we see that

$$\frac{Sa}{TVa} = \frac{Sb}{TVb} = \mathrm{a}, \text{ suppose.}$$

If, then, we separate q into its scalar and vector parts, thus

$$q = r+\rho,$$

the given equation becomes

$$(\mathrm{a}+\alpha)(r+\rho) = (r+\rho)(\mathrm{a}+\beta). \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Multiplying out we have

$$r(\alpha-\beta) = \rho\beta-\alpha\rho,$$

which gives $$S(\alpha-\beta)\rho = 0,$$
and therefore $$\rho = V\gamma(\alpha-\beta),$$
where γ is an undetermined vector.

We have now

$$\begin{aligned} r(\alpha-\beta) &= \rho\beta-\alpha\rho \\ &= V\gamma(\alpha-\beta).\beta-\alpha V\gamma(\alpha-\beta) \\ &= \gamma(S\alpha\beta+1)-(\alpha-\beta)S\beta\gamma-\gamma(1+S\alpha\beta)-(\alpha-\beta)S\alpha\gamma \\ &= -(\alpha-\beta)S(\alpha+\beta)\gamma. \end{aligned}$$

Having thus determined r, we have

$$\begin{aligned} q &= -S(\alpha+\beta)\gamma+V\gamma(\alpha-\beta) \\ 2q &= -(\alpha+\beta)\gamma-\gamma(\alpha+\beta)+\gamma(\alpha-\beta)-(\alpha-\beta)\gamma \\ &= -2\alpha\gamma-2\gamma\beta. \end{aligned}$$

Here, of course, we may change the sign of γ, and write the solution of
$$aq = qb$$
in the form $$q = \alpha\gamma+\gamma\beta,$$
where γ is any vector, and
$$\alpha = UVa, \qquad \beta = UVb.$$

To verify this solution, we see by (1) that we require only to show that
$$\alpha q = q\beta.$$
But their common value is evidently
$$-\gamma+\alpha\gamma\beta.$$

It will be excellent practice for the student to represent the terms of this equation by versor-arcs, as in § 54, and to deduce the above solution from the diagram directly. He will find that the solution may thus be obtained almost intuitively.

184. No general method of solving quaternion equations of the second or higher degrees has yet been found; in fact, as will be shown immediately, even those of the second degree involve (in their most general form) algebraic equations of the *sixteenth* degree. Hence, in the few remaining sections of this Chapter we shall confine ourselves to one or two of the simple forms for the treatment of which a definite process has been devised. But first, let us consider how many roots an equation of the second degree in an unknown quaternion must generally have.

If we substitute for the quaternion the expression

$$w+ix+jy+kz \quad (\S\ 80),$$

and treat the quaternion constants in the same way, we shall have (§ 80) four equations, generally of the second degree, to determine w, x, y, z. The number of roots will therefore be 2^4 or 16. And similar reasoning shows us that a quaternion equation of the mth degree has m^4 roots. It is easy to see, however, from some of the simple examples given above (§§ 175 –178, &c.) that, unless the given equation is equivalent to four scalar equations, the roots will contain one or more indeterminate quantities.

185. Hamilton has effected in a simple way the solution of the quadratic

$$q^2 = qa+b,$$

or the following, which is virtually the same (as we see by taking the conjugate of each side),

$$q^2 = aq+b.$$

He puts
$$q = \tfrac{1}{2}(a+w+\rho)$$

where w is a scalar, and ρ a vector.

Substituting this value, we get

$$a^2+(w+\rho)^2+2wa+a\rho+\rho a = 2(a^2+wa+\rho a)+4b,$$

$$\text{or} \qquad (w+\rho)^2+a\rho-\rho a = a^2+4b.$$

If we put $Va = \alpha$, $S(a^2+4b) = c$, $V(a^2+4b) = 2\gamma$, this becomes

$$(w+\rho)^2+2V\alpha\rho = c+2\gamma;$$

which, by equating separately the scalar and vector parts, may be broken up into the two equations

$$w^2+\rho^2 = c,$$
$$V(w+a)\rho = \gamma.$$

The latter of these can be solved for ρ by the process of § 156, or more simply by operating at once by $S.a$ which gives the value of $S(w+a)\rho$. If we substitute the resulting value of ρ in the former we obtain, as the reader may easily prove, the equation

$$(w^2-a^2)(w^4-cw^2+\gamma^2)-(Va\gamma)^2 = 0.$$

The solution of this scalar cubic gives six values of w, for each of which we find a value of ρ, and thence a value of q.

Hamilton shows (*Lectures*, p. 633) that only two of these values are real quaternions, the remaining four being biquaternions, and the other ten roots of the given equation being infinite.

Hamilton farther remarks that the above process leads, as the reader may easily see, to the solution of the two simultaneous equations

$$q+r = a,$$
$$qr = -b;$$

and he connects it also with the evaluation of certain continued fractions with quaternion constituents.

186. The equation

$$q^2 = aq+qb,$$

though apparently of the second degree, is easily reduced to the first degree by multiplying *by*, and *into*, q^{-1}, when it becomes

$$1 = q^{-1}a+bq^{-1},$$

and may be treated by the process of § 181.

187. The equation

$$\Sigma\, A_m q^m = aqb,$$

where a and b are given quaternions and A_m a scalar, gives easily

$$qaqb = aqbq. \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

This may be written

$$q(aqb) = (aqb)q;$$

and, by § 54, it is evident that the planes of q and aqb must coincide. A little farther consideration will show that in general we must have the planes of a, b, and q coincident. The solution of such equations thus becomes very easy, for the commutative law of multiplication is not violated (§ 54).

EXAMPLES TO CHAPTER V.

1. Solve the following equations:—

(*a*.) $V.\alpha\rho\beta = V.\alpha\gamma\beta.$

(*b*.) $\alpha\rho\beta\rho = \rho\alpha\rho\beta.$

(*c*.) $\alpha\rho + \rho\beta = \gamma.$

(*d*.) $S.\alpha\beta\rho + \beta S\alpha\rho - \alpha V\beta\rho = \gamma.$

(*e*.) $\rho + \alpha\rho\beta = \alpha\beta.$

Does the last of these impose any restriction on the generality of α and β?

2. Suppose $\rho = ix + jy + kz,$

and $\phi\rho = aiSi\rho + bjSj\rho + ckSk\rho;$

put into Cartesian cöordinates the following equations:—

(*a*.) $T\phi\rho = 1.$

(*b*.) $S\rho\phi^2\rho = -1.$

(*c*.) $S.\rho(\phi^2 - \rho^2)^{-1}\rho = -1.$

(*d*.) $T\rho = T.\phi U\rho.$

3. If λ, μ, ν be *any* three non-coplanar vectors, and

$$q = V\mu\nu . \phi\lambda + V\nu\lambda . \phi\mu + V\lambda\mu . \phi\nu,$$

show that q is necessarily divisible by $S.\lambda\mu\nu$.

Also show that the quotient is

$$m_2 - 2\epsilon,$$

where $V\epsilon\rho$ is the non-commutative part of $\phi\rho$.

Hamilton, *Elements*, p. 442.

4. Solve the simultaneous equations :—

$$(a.)\quad \left.\begin{aligned} S\alpha\rho &= 0, \\ S.\alpha\rho\phi\rho &= 0. \end{aligned}\right\}$$

$$(b.)\quad \left.\begin{aligned} S\alpha\rho &= 0, \\ S\rho\phi\rho &= 0. \end{aligned}\right\}$$

$$(c.)\quad \left.\begin{aligned} S\alpha\rho &= 0, \\ S.\alpha\iota\rho\kappa\rho &= 0. \end{aligned}\right\}$$

5. If $\qquad \phi\rho = \Sigma\beta S\alpha\rho + Vr\rho,$

where r is a given quaternion, show that

$$m = \Sigma(S.\alpha_1\alpha_2\alpha_3 S.\beta_3\beta_2\beta_1) + \Sigma S(rV\alpha_1\alpha_2 . V\beta_2\beta_1) + Sr\Sigma S.\alpha\beta r - \Sigma(S\alpha rS\beta r) + SrTr^2,$$

and

$$m\phi^{-1}\sigma = \Sigma(V\alpha_1\alpha_2 S.\beta_2\beta_1\sigma) + \Sigma V.\alpha V(V\beta\sigma.r) + V\sigma rSr - VrS\sigma r.$$

Lectures, p. 561.

6. If $[pq]$ denote $pq - qp$,

(pq) ... $S.p\,[qr]$,

$[pqr]$ ... $(pqr) + [rq]Sp + [pr]Sq + [qp]Sr$,

and $(pqrs)$ ... $S.p[qrs]$,

show that the following relations exist among any five quaternions

$$0 = p(qrst) + q(rstp) + r(stpq) + s(tpqr) + t(pqrs),$$

and $\qquad q(prst) = [rst]Spq - [stp]Srq + [tpr]Ssq - [prs]Stq.$

Elements, p. 492.

7. Show that if ϕ, ψ be any linear and vector functions, and α, β, γ rectangular unit-vectors, the vector

$$\theta = V(\phi\alpha\psi\alpha + \phi\beta\psi\beta + \phi\gamma\psi\gamma)$$

is an invariant.

$$\text{If} \qquad \phi\rho = \Sigma\eta S\zeta\rho,$$

$$\text{and} \qquad \psi\rho = \Sigma\eta_1 S\zeta_1\rho,$$

show that this invariant may be expressed as

$$-\Sigma V\eta\psi\zeta \text{ or } \Sigma V\eta_1\phi\zeta_1.$$

Show also that $\qquad \phi\psi\rho - \psi\phi\rho = V\theta\rho.$

8. Show that if

$$\phi\rho = \alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho,$$

where α, β, γ are any three vectors, then

$$-S^2.\alpha\beta\gamma.\phi^{-1}\rho = \alpha_1 S\alpha_1\rho + \beta_1 S\beta_1\rho + \gamma_1 S\gamma_1\rho,$$

where $\alpha_1 = V\beta\gamma$, &c.

9. Show that any self-conjugate linear and vector function may in general be expressed in terms of two given ones, the expression involving terms of the second order.

Show also that we may write

$$\phi + z = a(\varpi + x)^2 + b(\varpi + x)(\omega + y) + c(\omega + y)^2,$$

where a, b, c, x, y, z are scalars, and ϖ and ω the two given functions.

10. Solve the equations:—

(a.) $q^2 = 5qi + 10j.$

(b.) $q^2 = 2q + i.$

(c.) $qaq = bq + c.$

(d.) $aq = qr = rb.$

CHAPTER VI.

GEOMETRY OF THE STRAIGHT LINE AND PLANE.

188. HAVING, in the five preceding Chapters, given a brief exposition of the theory and properties of quaternions, we intend to devote the rest of the work to examples of their practical application, commencing, of course, with the simplest curve and surface, the straight line and the plane. In this and the remaining Chapters of the work a few of the earlier examples will be wrought out in their fullest detail, with a reference to the first five whenever a transformation occurs; but, as each Chapter proceeds, superfluous steps will be gradually omitted, until in the later examples the full value of the quaternion processes is exhibited.

189. Before proceeding to the proper business of the Chapter we make a digression in order to give a few instances of applications to ordinary plane geometry. These the student may multiply indefinitely with great ease.

(*a*.) *Euclid*, I. 5. Let α and β be the vector sides of an isosceles triangle; $\beta-\alpha$ is the base, and

$$T\alpha = T\beta.$$

The proposition will evidently be proved if we show that

$$\alpha(\alpha-\beta)^{-1} = K\beta(\beta-\alpha)^{-1} \quad (\S\ 52).$$

This gives $$\alpha(\alpha-\beta)^{-1} = (\beta-\alpha)^{-1}\beta,$$

or $$(\beta-\alpha)\alpha = \beta(\alpha-\beta),$$

or $$-\alpha^2 = -\beta^2.$$

(*b.*) *Euclid,* I. 32. Let *ABC* be the triangle, and let

$$U\frac{\overline{AC}}{\overline{AB}} = \gamma^l,$$

where γ is a unit-vector perpendicular to the plane of the triangle. If $l = 1$, the angle *CAB* is a right angle (§ 74). Hence $A = l\frac{\pi}{2}$ (§ 74). Let $B = m\frac{\pi}{2}$, $C = n\frac{\pi}{2}$. We have

$$U\overline{AC} = \gamma^l\, U\overline{AB},$$

$$U\overline{CB} = \gamma^n\, U\overline{CA},$$

$$U\overline{BA} = \gamma^m U\overline{BC}.$$

Hence $$U\overline{BA} = \gamma^m.\gamma^n.\gamma^l U\overline{AB},$$

or $$-1 = \gamma^{l+m+n}.$$

That is $$l+m+n = 2,$$

or $$A+B+C = \pi.$$

This is, properly speaking, Legendre's proof; and might have been given in a far shorter form than that above. In fact we have for any three vectors,

$$U.\frac{\alpha}{\beta}\frac{\beta}{\gamma}\frac{\gamma}{\alpha} = 1 \quad (\S\ 60),$$

which contains Euclid's proposition as a mere particular case.

(*c.*) *Euclid,* I. 35. Let β be the common vector-base of the parallelograms, α the conterminous vector-side of any one of them. For any other the vector-side is $\alpha + x\beta$ (§ 28), and the proposition appears as

$$TV\beta(\alpha+x\beta) = TV\beta\alpha \quad (\S\S\ 96,\ 98),$$

which is obviously true.

(*d.*) In the base of a triangle find the point from which lines, drawn parallel to the sides and limited by them, are equal.

If α, β be the sides, any point in the base has the vector

$$\rho = (1-x)\alpha + x\beta.$$

For the required point

$$(1-x)T\alpha = xT\beta$$

which determines x.

Hence the point lies on the line

$$\rho = y(U\alpha + U\beta)$$

which bisects the vertical angle of the triangle.

This is not the only solution, for we should have written

$$T(1-x)T\alpha = TxT\beta,$$

instead of the less general form above *which tacitly assumes that* $1-x$ *and* x *are positive.* We leave this to the student.

(*e.*) If perpendiculars be erected outwards at the middle points of the sides of a triangle, each being proportional to the corresponding side, the mean point of the triangle formed by their extremities coincides with that of the original triangle. Find the ratio of each perpendicular to half the corresponding side of the old triangle that the new triangle may be equilateral.

Let 2α, 2β, and $2(\alpha+\beta)$ be the vector-sides of the triangle, i a unit-vector perpendicular to its plane, e the ratio in question. The vectors of the corners of the new triangle are (taking the corner opposite to 2β as origin)

$$\begin{aligned}\rho_1 &= \alpha + ei\alpha,\\ \rho_2 &= 2\alpha + \beta + ei\beta,\\ \rho_3 &= \alpha + \beta - ei(\alpha+\beta).\end{aligned}$$

From these

$$\tfrac{1}{3}(\rho_1+\rho_2+\rho_3) = \tfrac{1}{3}(4\alpha+2\beta) = \tfrac{1}{3}(2\alpha+2(\alpha+\beta)),$$

which proves the first part of the proposition.

For the second part, we must have

$$T(\rho_2-\rho_1) = T(\rho_3-\rho_2) = T(\rho_1-\rho_3).$$

Substituting, expanding, and erasing terms common to all, the student will easily find $3e^2 = 1.$

Hence, if equilateral triangles be described on the sides of any triangle, their mean points form an equilateral triangle.

190. Such applications of quaternions as those just made are of course legitimate, but they are not always profitable. In fact, when applied to plane problems, quaternions often degenerate into mere scalars, and become (§ 33) Cartesian cöordinates of some kind, so that nothing is gained (though nothing is lost) by their use. Before leaving this class of questions we take, as an additional example, the investigation of some properties of the ellipse.

191. We have already seen (§ 31 (*k*)) that the equation

$$\rho = \alpha\cos\theta+\beta\sin\theta$$

represents an ellipse, θ being a scalar which may have any value. Hence, for the vector-tangent at the extremity of ρ we have

$$\frac{d\rho}{d\theta} = -\alpha\sin\theta+\beta\cos\theta,$$

which is easily seen to be the value of ρ when θ is increased by $\frac{\pi}{2}$. Thus it appears that any two values of ρ, for which θ differs by $\frac{\pi}{2}$, are conjugate diameters. The area of the parallelogram circumscribed to the ellipse and touching it at the extremities of these diameters is, therefore, by § 96,

$$\begin{aligned} 4\,TV\rho\frac{d\rho}{d\theta} &= 4\,TV(\alpha\cos\theta+\beta\sin\theta)(-\alpha\sin\theta+\beta\cos\theta) \\ &= 4\,TV\alpha\beta, \end{aligned}$$

a constant, as is well known.

192. For equal conjugate diameters we must have

$$T(\alpha\cos\theta+\beta\sin\theta) = T(-\alpha\sin\theta+\beta\cos\theta),$$

or $$(\alpha^2-\beta^2)(\cos^2\theta-\sin^2\theta)+4S\alpha\beta\cos\theta\sin\theta = 0,$$

or $$\tan 2\theta = -\frac{\alpha^2-\beta^2}{2S\alpha\beta}.$$

The square of the common length of these diameters is of course

$$-\frac{\alpha^2+\beta^2}{2},$$

because we see at once from § 191 that the sum of the squares of conjugate diameters is constant.

193. The maximum or minimum of ρ is thus found;

$$\frac{dT\rho}{d\theta} = -\frac{1}{T\rho}S\rho\frac{d\rho}{d\theta}.$$

$$= -\frac{1}{T\rho}(-(\alpha^2-\beta^2)\cos\theta\sin\theta+S\alpha\beta(\cos^2\theta-\sin^2\theta)).$$

For a maximum or minimum this must vanish, hence

$$\tan 2\theta = \frac{2S\alpha\beta}{\alpha^2-\beta^2},$$

and therefore the longest and shortest diameters are equally inclined to each of the equal conjugate diameters. Hence, also, they are at right angles to each other.

[The student must carefully notice that here we put $\frac{dT\rho}{d\theta} = 0$, and not $\frac{d\rho}{d\theta} = 0$. A little reflection will show him that the latter equation involves an absurdity.]

194. Suppose for a moment α and β to be the greatest and least semidiameters. Then the equations of any two tangent-lines are

$$\rho = \alpha\cos\theta + \beta\sin\theta + x(-\alpha\sin\theta + \beta\cos\theta),$$

$$\rho = \alpha\cos\theta_1 + \beta\sin\theta_1 + x_1(-\alpha\sin\theta_1 + \beta\cos\theta_1).$$

If these tangent lines be at right angles to each other

$$S(-\alpha\sin\theta+\beta\cos\theta)(-\alpha\sin\theta_1+\beta\cos\theta_1) = 0,$$

or $$\alpha^2\sin\theta\sin\theta_1+\beta^2\cos\theta\cos\theta_1 = 0.$$

Also, for their point of intersection we have, by comparing coefficients of α, β in the above values of ρ,

$$\cos\theta-x\sin\theta = \cos\theta_1-x_1\sin\theta_1,$$

$$\sin\theta+x\cos\theta = \sin\theta_1+x_1\cos\theta_1.$$

Determining x_1 from these equations, we easily find

$$T\rho^2 = -(\alpha^2+\beta^2),$$

the equation of a circle; if we take account of the above relation between θ and θ_1.

Also, as the equations above give $x = -x_1$, the tangents are equal multiples of the diameters parallel to them; so that the line joining the points of contact is parallel to that joining the extremities of these diameters.

195. Finally, when the tangents

$$\rho = \alpha\cos\theta\ +\beta\sin\theta\ +x\ (-\alpha\sin\theta\ +\beta\cos\theta),$$

$$\rho = \alpha\cos\theta_1+\beta\sin\theta_1+x_1(-\alpha\sin\theta_1+\beta\cos\theta_1),$$

meet in a given point

$$\rho = a\alpha+b\beta,$$

we have

$$a = \cos\theta-x\sin\theta = \cos\theta_1-x_1\sin\theta_1,$$

$$b = \sin\theta+x\cos\theta = \sin\theta_1+x_1\cos\theta_1.$$

Hence $$x^2 = a^2+b^2-1 = x_1^2$$

and $$a\cos\theta+b\sin\theta = 1 = a\cos\theta_1+b\sin\theta_1$$

determine the values of θ and x for the directions and lengths of the two tangents. The equation of the chord of contact is

$$\rho = y(\alpha\cos\theta+\beta\sin\theta)+(1-y)(\alpha\cos\theta_1+\beta\sin\theta_1).$$

If this pass through the point

$$\rho = p\alpha+q\beta,$$

we have
$$p = y\cos\theta + (1-y)\cos\theta_1,$$
$$q = y\sin\theta + (1-y)\sin\theta_1,$$
from which, by the equations which determine θ and θ_1, we get
$$ap + bq = y + 1 - y = 1.$$
Thus if either a and b, or p and q, be given, a linear relation connects the others. This, by § 30, gives all the ordinary properties of poles and polars.

196. Although, in §§ 28–30, we have already given some of the equations of the line and plane, these were adduced merely for their applications to anharmonic coördinates and transversals; and not for investigations of a higher order. Now that we are prepared to determine the lengths and inclinations of lines we may investigate these and other similar forms anew.

197. The equation of the indefinite line drawn through the origin O, of which the vector $\overline{OA}$, $= \alpha$, forms a part, is evidently
$$\rho = x\alpha,$$
or
$$\rho \parallel \alpha,$$
or
$$V\alpha\rho = 0,$$
or
$$U\rho = U\alpha;$$
the essential characteristic of these equations being that they are linear, and involve *one* indeterminate scalar in the value of ρ.

We may put this perhaps more clearly if we take any two vectors, β, γ, which, along with α, form a non-coplanar system. Operating with $S.V\alpha\beta$ and $S.V\alpha\gamma$ upon any of the preceding equations, we get
$$\left.\begin{array}{r} S.\alpha\beta\rho = 0, \\ \text{and} \qquad S.\alpha\gamma\rho = 0. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$
Separately, these are the equations of the planes containing α, β, and α, γ; together, of course, they denote the line of intersection.

198. Conversely, to solve equations (1), or to find ρ in terms of known quantities, we see that they may be written

$$\left.\begin{aligned} S.\rho V\alpha\beta &= 0, \\ S.\rho V\alpha\gamma &= 0, \end{aligned}\right\}$$

so that ρ is perpendicular to $V\alpha\beta$ and $V\alpha\gamma$, and is therefore parallel to the vector of their product. That is,

$$\begin{aligned} \rho &\parallel V.V\alpha\beta V\alpha\gamma, \\ &\parallel -\alpha S.\alpha\beta\gamma, \\ \text{or} \qquad \rho &= x\alpha. \end{aligned}$$

199. By putting $\rho-\beta$ for ρ we change the origin to a point B where $\overline{OB} = -\beta$, or $\overline{BO} = \beta$; so that the equation of a line parallel to α, and passing through the extremity of a vector β drawn from the origin, is

$$\begin{aligned} \rho-\beta &= x\alpha, \\ \text{or} \qquad \rho &= \beta+x\alpha. \end{aligned}$$

Of course any two parallel lines may be represented as

$$\begin{aligned} \rho &= \beta + x\alpha, \\ \rho &= \beta_1+x_1\alpha; \\ \text{or} \qquad V\alpha(\rho-\beta) &= 0, \\ V\alpha(\rho-\beta_1) &= 0. \end{aligned}$$

200. The equation of a line, drawn through the extremity of β, and meeting α perpendicularly, is thus found. Suppose it to be parallel to γ, its equation is

$$\rho = \beta+x\gamma.$$

To determine γ we know, *first*, that it is perpendicular to α, which gives

$$S\alpha\gamma = 0.$$

Secondly, α, β, and γ are in one plane, which gives

$$S.\alpha\beta\gamma = 0.$$

These two equations give

$$\gamma \parallel V.\alpha V\alpha\beta,$$

whence we have

$$\rho = \beta + x\alpha V\alpha\beta.$$

This might have been obtained in many other ways; for instance, we see at once that

$$\beta = \alpha^{-1}\alpha\beta = \alpha^{-1}S\alpha\beta + \alpha^{-1}V\alpha\beta.$$

This shows that $\alpha^{-1}V\alpha\beta$ (which is evidently perpendicular to α) is coplanar with α and β, and is therefore the direction of the required line; so that its equation is

$$\rho = \beta + y\alpha^{-1}V\alpha\beta,$$

the same as before if we put $-\frac{y}{T\alpha^2}$ for x.

201. By means of the last investigation we see that

$$-\alpha^{-1}V\alpha\beta$$

is the vector perpendicular drawn from the extremity of β to the line

$$\rho = x\alpha.$$

Changing the origin, we see that

$$-\alpha^{-1}V\alpha(\beta-\gamma)$$

is the vector perpendicular from the extremity of β upon the line

$$\rho = \gamma + x\alpha.$$

202. The vector joining B (where $\overline{OB} = \beta$) with any point in

$$\rho = \gamma + x\alpha$$

is

$$\gamma + x\alpha - \beta.$$

Its length is least when

$$dT(\gamma + x\alpha - \beta) = 0,$$

or

$$S\alpha(\gamma + x\alpha - \beta) = 0,$$

i. e. when it is perpendicular to α.

The last equation gives

$$x\alpha^2 + S\alpha(\gamma - \beta) = 0,$$

or

$$x\alpha = -\alpha^{-1}S\alpha(\gamma - \beta).$$

Hence the vector perpendicular is

$$\gamma-\beta-\alpha^{-1}S\alpha(\gamma-\beta),$$

$$\text{or} \qquad \alpha^{-1}V\alpha(\gamma-\beta) = -\alpha^{-1}V\alpha(\beta-\gamma),$$

which agrees with the result of last section.

203. To find the shortest vector distance between two lines

$$\rho = \beta + x\alpha,$$

$$\text{and} \qquad \rho_1 = \beta_1 + x_1\alpha_1\,;$$

we must put $$dT(\rho-\rho_1) = 0,$$

$$\text{or} \qquad S(\rho-\rho_1)(d\rho-d\rho_1) = 0,$$

$$\text{or} \qquad S(\rho-\rho_1)(\alpha dx + \alpha_1 dx_1) = 0.$$

Since x and x_1 are independent, this breaks up into the two conditions

$$S\alpha\,(\rho-\rho_1) = 0,$$

$$S\alpha_1(\rho-\rho_1) = 0\,;$$

proving the well-known truth that the required line is perpendicular to each of the given lines.

Hence it is parallel to $V\alpha\alpha_1$, and therefore we have

$$\rho-\rho_1 = \beta + x\alpha - \beta_1 - x_1\alpha_1 = yV\alpha\alpha_1. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Operate by $S.\alpha\alpha_1$ and we get

$$S.\alpha\alpha_1(\beta-\beta_1) = y(V\alpha\alpha_1)^2.$$

This determines y, and the shortest distance required is

$$T(\rho-\rho_1) = T(yV\alpha\alpha_1) = \frac{TS.\alpha\alpha_1(\beta-\beta_1)}{TV\alpha\alpha_1} = TS.(UV\alpha\alpha_1)(\beta-\beta_1).$$

[Note. In the two last expressions T before S is simply inserted to ensure that the length be positive. If

$$S.\alpha\alpha_1(\beta-\beta_1) \text{ be negative,}$$

then (§ 89) $$S.\alpha_1\alpha(\beta-\beta_1) \text{ is positive.}$$

If we omit the T we must use in the text that one of these two expressions which is positive.]

To find the extremities of this shortest distance, we must operate on (1) with $S.\alpha$ and $S.\alpha_1$. We thus obtain two equations, which determine x and x_1, as y is already known.

A somewhat different mode of treating this problem will be discussed presently.

204. The equation

$$S\alpha\rho = 0$$

imposes on ρ the sole condition of being perpendicular to α; and therefore, being satisfied by the vector drawn from the origin to any point in a plane through the origin and perpendicular to α, is the equation of that plane.

To find this equation by a direct process similar to that usually employed in cöordinate geometry, we may remark that, by § 29, we may write

$$\rho = x\beta + y\gamma,$$

where β and γ are any two vectors perpendicular to α. In this form the equation contains two indeterminates, and is often useful; but it is more usual to eliminate them, which may be done at once by operating by $S.\alpha$, when we obtain the equation first written.

It may also be written, by eliminating one of the indeterminates only, as

$$V\beta\rho = y\alpha,$$

where the form of the equation shows that $S\alpha\beta = 0$.

205. Similarly we see that

$$S\alpha(\rho - \beta) = 0$$

represents a plane drawn through the extremity of β and perpendicular to α. This, of course, may, like the last, be put into various equivalent forms.

206. The line of intersection of the two planes

$$\left.\begin{aligned} &S.\alpha\,(\rho-\beta) = 0, \\ \text{and}\qquad &S.\alpha_1(\rho-\beta_1) = 0, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

contains all points whose value of ρ satisfies both conditions. But we may write (§ 92), since α, α_1, and $V\alpha\alpha_1$ are not coplanar,

$$\rho S.\alpha\alpha_1 V\alpha\alpha_1 = V\alpha\alpha_1 S.\alpha\alpha_1\rho + V.\alpha_1 V\alpha\alpha_1 S\alpha\rho + V.V(\alpha\alpha_1)\alpha S\alpha_1\rho,$$

or, by the given equations,

$$-\rho T^2 V\alpha\alpha_1 = V.\alpha_1 V\alpha\alpha_1 S\alpha\beta + V.V(\alpha\alpha_1)\alpha S\alpha_1\beta_1 + xV\alpha\alpha_1, \qquad (2)$$

where x, a scalar indeterminate, is put for $S.\alpha\alpha_1\rho$ which may have any value. In practice, however, the two definite given scalar equations are generally more useful than the partially indeterminate vector-form which we have derived from them.

When both planes pass through the origin we have $\beta = \beta_1 = 0$, and obtain at once

$$\rho = xV\alpha\alpha_1$$

as the equation of the line of intersection.

207. The plane passing through the origin, and through the line of intersection of the two planes (1), is easily seen to have the equation

$$S\alpha_1\beta_1 S\alpha\rho - S\alpha\beta S\alpha_1\rho = 0,$$

or

$$S(\alpha S\alpha_1\beta_1 - \alpha_1 S\alpha\beta)\rho = 0.$$

For this is evidently the equation of a plane passing through the origin. And, if ρ be such that

$$S\alpha\rho = S\alpha\beta,$$

we also have

$$S\alpha_1\rho = S\alpha_1\beta_1,$$

which are equations (1).

Hence we see that the vector

$$\alpha S\alpha_1\beta_1 - \alpha_1 S\alpha\beta$$

is perpendicular to the vector-line of intersection (2) of the two planes (1), and to every vector joining the origin with a point in that line.

The student may verify these statements as an exercise.

208. To find the vector-perpendicular from the extremity of β on the plane

$$S\alpha\rho = 0,$$

we must note that it is necessarily parallel to α, and hence that the corresponding value of ρ is

$$\rho = \beta + x\alpha,$$

where $x\alpha$ is the vector perpendicular in question.

Hence $$S\alpha(\beta + x\alpha) = 0,$$

which gives $$x\alpha^2 = -S\alpha\beta,$$

or $$x\alpha = -\alpha^{-1}S\alpha\beta.$$

Similarly the vector-perpendicular from the extremity of β on the plane $$S\alpha(\rho - \gamma) = 0$$

may easily be shown to be

$$-\alpha^{-1}S\alpha(\beta - \gamma).$$

209. The equation of the plane which passes through the extremities of α, β, γ may be thus found. If ρ be the vector of any point in it, $\rho - \alpha$, $\alpha - \beta$, and $\beta - \gamma$ lie in the plane, and therefore (§ 101)

$$S.(\rho - \alpha)(\alpha - \beta)(\beta - \gamma) = 0,$$

or $$S\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) - S.\alpha\beta\gamma = 0.$$

Hence, if $$\delta = x(V\alpha\beta + V\beta\gamma + V\gamma\alpha)$$

be the vector-perpendicular from the origin on the plane containing the extremities of α, β, γ, we have

$$\delta = (V\alpha\beta + V\beta\gamma + V\gamma\alpha)^{-1}S.\alpha\beta\gamma.$$

From this formula, whose interpretation is easy, many curious properties of a tetrahedron may be deduced by the reader.

210. Taking any two lines whose equations are

$$\rho = \beta + x\alpha,$$

$$\rho = \beta_1 + x_1\alpha_1,$$

we see that $$S.\alpha\alpha_1(\rho - \delta) = 0$$

is the equation of a plane parallel to both. *Which* plane, of course, depends on the value of δ.

Now if $\delta = \beta$, the plane contains the first line; if $\delta = \beta_1$, the second.

Hence, if $yVaa_1$ be the shortest vector distance between the lines, we have

$$S.aa_1(\beta-\beta_1-yVaa_1) = 0,$$

or $$T(yVaa_1) = TS.(\beta-\beta_1)UVaa_1,$$

the result of § 203.

211. Find the equation of the plane, passing through the origin, which makes equal angles with three given lines. Also find the angles in question.

Let a, β, γ be unit-vectors in the directions of the lines, and let the equation of the plane be

$$S\delta\rho = 0.$$

Then we have evidently

$$Sa\delta = S\beta\delta = S\gamma\delta = x, \text{ suppose,}$$

where $$-\frac{x}{T\delta}$$

is the sine of each of the required angles.

But (§ 92) we have

$$\delta S.a\beta\gamma = x(Va\beta + V\beta\gamma + V\gamma a).$$

Hence $$S.\rho(Va\beta + V\beta\gamma + V\gamma a) = 0$$

is the required equation; and the required sine is

$$\frac{S.a\beta\gamma}{T(Va\beta + V\beta\gamma + V\gamma a)}.$$

212. Find the locus of the middle points of a series of straight lines, each parallel to a given plane and having its extremities in two fixed lines.

Let $$S\gamma\rho = 0$$

be the plane, and

$$\rho = \beta + xa, \qquad \rho = \beta_1 + x_1 a_1,$$

the fixed lines. Also let x and x_1 correspond to the extremities of one of the variable lines, ϖ being the vector of its middle point. Then, obviously,

$$2\varpi = \beta+x\alpha+\beta_1+x_1\alpha_1.$$

Also $$S\gamma(\beta-\beta_1+x\alpha-x_1\alpha_1) = 0.$$

This gives a linear relation between x and x_1, so that, if we substitute for x_1 in the preceding equation, we obtain a result of the form

$$\varpi = \delta+x\epsilon,$$

where δ and ϵ are known vectors. The required locus is, therefore, a straight line.

213. Three planes meet in a point, and through the line of intersection of each pair a plane is drawn perpendicular to the third; prove that, in general, these planes pass through the same line.

Let the point be taken as origin, and let the equations of the planes be

$$S\alpha\rho = 0, \qquad S\beta\rho = 0, \qquad S\gamma\rho = 0.$$

The line of intersection of the first two is $\parallel V\alpha\beta$, and therefore the normal to the first of the new planes is

$$V.\gamma V\alpha\beta.$$

Hence the equation of this plane is

$$S.\rho V.\gamma V\alpha\beta = 0,$$

or $$S\beta\rho S\alpha\gamma-S\alpha\rho S\beta\gamma = 0,$$

and those of the other two planes may be easily formed from this by cyclical permutation of α, β, γ.

We see at once that any two of these equations give the third by addition or subtraction, which is the proof of the theorem.

214. Given any number of points A, B, C, &c., whose vectors (from the origin) are α_1, α_2, α_3, &c., find the plane through the origin for which the sum of the squares of the perpendiculars let fall upon it from these points is a maximum or minimum.

Let $$S\varpi\rho = 0$$
be the required equation, with the condition (evidently allowable)

$$T\varpi = 1.$$

The perpendiculars are (§ 208) $-\varpi^{-1}S\varpi\alpha_1$, &c.

Hence $$\Sigma\, S^2\,\varpi\alpha$$
is a maximum. This gives

$$\Sigma . S\varpi\alpha S\alpha d\varpi = 0\,;$$

and the condition that ϖ is a unit-vector gives

$$S\varpi d\varpi = 0.$$

Hence, as $d\varpi$ may have any of an infinite number of values, these equations cannot be consistent unless

$$\Sigma . \alpha S\alpha\varpi = x\varpi,$$
where x is a scalar.

The values of α are known, so that if we put

$$\Sigma . \alpha S\alpha\varpi = \phi\varpi,$$

ϕ is a given self-conjugate linear and vector function, and therefore x has three values (g_1, g_2, g_3, § 164) which correspond to three mutually perpendicular values of ϖ. For one of these there is a maximum, for another a minimum, for the third a maximum-minimum, in the most general case when g_1, g_2, g_3 are all different.

215. The following beautiful problem is due to Maccullagh. Of a system of three rectangular vectors, passing through the origin, two lie on given planes, find the locus of the third.

Let the rectangular vectors be ϖ, ρ, σ. Then by the conditions of the problem

$$S\varpi\rho = S\rho\sigma = S\sigma\varpi = 0,$$

$$\text{and} \qquad S\alpha\varpi = 0, \qquad S\beta\rho = 0.$$

The solution depends on the elimination of ρ and ϖ among these five equations. [This would, in general, be impossible, as ρ

and ϖ between them involve six unknown scalars; but, as the tensors are (by the very form of the equations) not involved, the five given equations serve to eliminate the four unknown scalars which are really involved. Formally to complete the requisite number of equations we might write

$$T\varpi = a, \qquad T\rho = b,$$

but a and b may have any values whatever.]

From $\quad S\alpha\varpi = 0, \qquad S\sigma\varpi = 0,$

we have $$\varpi = xV\alpha\sigma.$$

Similarly, from $\quad S\beta\rho = 0, \qquad S\sigma\rho = 0,$

we have $$\rho = yV\beta\sigma.$$

Substitute in the remaining equation

$$S\varpi\rho = 0,$$

and we have

$$S.V\alpha\sigma V\beta\sigma = 0,$$

or $$S\alpha\sigma S\beta\sigma - \sigma^2 S\alpha\beta = 0,$$

the required equation. As will be seen in next Chapter, this is a cone of the second order whose circular sections are perpendicular to α and β. [The disappearance of x and y in the elimination instructively illustrates the note above.]

EXAMPLES TO CHAPTER VI.

1. What propositions of Euclid are proved by the mere *form* of the equation

$$\rho = (1-x)\alpha + x\beta,$$

which denotes the line joining any two points in space?

2. Show that the chord of contact, of tangents to a parabola which meet at right angles, passes through a fixed point.

3. Prove the chief properties of the circle (as in *Euclid*, III) from the equation

$$\rho = \alpha\cos\theta + \beta\sin\theta;$$

where $T\alpha = T\beta$, and $S\alpha\beta = 0$.

4. What locus is represented by the equation

$$S^2\alpha\rho + \rho^2 = 0,$$

where $T\alpha = 1$?

5. What is the condition that the lines

$$V\alpha\rho = \beta, \quad V\alpha_1\rho = \beta_1,$$

intersect? If this is not satisfied, what is the shortest distance between them?

6. Find the equation of the plane which contains the two parallel lines

$$V\alpha(\rho-\beta) = 0, \qquad V\alpha(\rho-\beta_1) = 0.$$

7. Find the equation of the plane which contains

$$V\alpha(\rho-\beta) = 0,$$

and is perpendicular to

$$S\gamma\rho = 0.$$

8. Find the equation of a straight line passing through a given point, and making a given angle with a given plane.

Hence form the general equation of a right cone.

9. What conditions must be satisfied with regard to a number of given lines in space that it may be possible to draw through each of them a plane in such a way that these planes may intersect in a common line?

10. Find the equation of the locus of a point the sum of the squares of whose distances from a number of given planes is constant.

11. Substitute "lines" for "planes" in (10).

12. Find the equation of the plane which bisects, at right angles, the shortest distance between two given lines.

Find the locus of a point in this plane which is equidistant from the given lines.

13. Find the conditions that the simultaneous equations

$$S\alpha\rho = a, \qquad S\beta\rho = b, \qquad S\gamma\rho = c,$$

may represent a line, and not a point.

14. What is represented by the equations

$$(S\alpha\rho)^2 = (S\beta\rho)^2 = (S\gamma\rho)^2,$$

where α, β, γ are any three vectors?

15. Find the equation of the plane which passes through two given points and makes a given angle with a given plane.

16. Find the area of the triangle whose corners have the vectors α, β, γ.

Hence form the equation of a circular cylinder whose axis and radius are given.

17. (Hamilton, *Bishop Law's Premium Ex.*, 1858).

(*a*.) Assign some of the transformations of the expression

$$\frac{V\alpha\beta}{\beta-\alpha},$$

where α and β are the vectors of two given points A and B.

(*b*.) The expression represents the vector γ, or $\overline{OC}$, of a point C in the straight line AB.

(*c*.) Assign the position of this point C.

18. (*Ibid.*)

(*a*.) If $\alpha, \beta, \gamma, \delta$ be the vectors of four points A, B, C, D, what is the condition for those points being in one plane?

(*b*.) When these four vectors from one origin do not thus terminate upon one plane, what is the expression for the volume of the pyramid, of which the four points are the corners?

(*c*.) Express the perpendicular δ let fall from the origin O on the plane ABC, in terms of α, β, γ.

19. Find the locus of a point equidistant from the three planes

$$S\alpha\rho = 0, \qquad S\beta\rho = 0, \qquad S\gamma\rho = 0.$$

20. If three mutually perpendicular vectors be drawn from a point to a plane, the sum of the reciprocals of the squares of their lengths is independent of their directions.

21. Find the general form of the equation of a plane from the condition (which is to be assumed as a definition) that any two planes intersect in a single straight line.

CHAPTER VII.

THE SPHERE AND CYCLIC CONE.

216. AFTER that of the plane the equations next in order of simplicity are those of the sphere, and of the cone of the second order. To these we devote a short Chapter as a valuable preparation for the study of surfaces of the second order in general.

217. The equation

$$T\rho = Ta,$$

$$\text{or} \qquad \rho^2 = a^2,$$

denotes that the length of ρ is the same as that of a given vector a, and therefore belongs to a sphere of radius Ta whose centre is the origin. In § 107 several transformations of this equation were obtained, some of which we will repeat here with their interpretations. Thus

$$S(\rho+a)(\rho-a) = 0$$

shows that the chords drawn from any point on the sphere to the extremities of a diameter (whose vectors are a and $-a$) are at right angles to each other.

$$T(\rho+a)(\rho-a) = 2\,TVa\rho$$

shows that the rectangle under these chords is four times the area of the triangle two of whose sides are a and ρ.

$$\rho = (\rho+a)^{-1}a(\rho+a) \quad (\text{see § 105})$$

shows that the angle at the centre in any circle is double that at the circumference standing on the same arc. All these are easy consequences of the processes already explained for the interpretation of quaternion expressions.

218. If the centre of the sphere be at the extremity of α the equation may be written

$$T(\rho-\alpha) = T\beta,$$

which is the most general form.

$$\text{If} \qquad T\alpha = T\beta,$$

$$\text{or} \qquad \alpha^2 = \beta^2,$$

in which case the origin is a point on the surface of the sphere, this becomes

$$\rho^2-2S\alpha\rho = 0.$$

From this, in the form

$$S\rho(\rho-2\alpha) = 0$$

another proof that the angle in a semicircle is a right angle is derived at once.

219. The converse problem is—Find the locus of the feet of perpendiculars let fall from a given point ($\rho = \beta$) on planes passing through the origin.

Let $$S\alpha\rho = 0$$

be one of the planes, then (§ 208) the vector-perpendicular is

$$-\alpha^{-1}S\alpha\beta,$$

and, for the locus of its foot,

$$\begin{aligned}\rho &= \beta-\alpha^{-1}S\alpha\beta \\ &= \alpha^{-1}V\alpha\beta.\end{aligned}$$

[This is an example of a peculiar form in which quaternions sometimes give us the equation of a surface. The equation is a vector one, or equivalent to three scalar equations; but it involves the undetermined vector α in such a way as to be equivalent to only two indeterminates (as the tensor of α is evidently not involved). To put the equation in a more immediately interpretable form, α must be eliminated, and the remarks just made show this to be possible.]

Now $$(\rho-\beta)^2 = \alpha^{-2}S^2\alpha\beta,$$

and (operating by $S.\beta$)

$$S\beta\rho-\beta^2 = -\alpha^{-2}S^2\alpha\beta.$$

Adding these equations, we get

$$\rho^2-S\beta\rho = 0,$$

or $$T\left(\rho-\frac{\beta}{2}\right) = T\frac{\beta}{2},$$

so that, as is evident, the locus is the sphere of which β is a diameter.

220. To find the intersection of the two spheres

$$T(\rho-\alpha) = T\beta,$$

and $$T(\rho-\alpha_1) = T\beta_1,$$

square the equations, and subtract, and we have

$$2S(\alpha-\alpha_1)\rho = \alpha^2-\alpha_1^2-(\beta^2-\beta_1^2),$$

which is the equation of a plane, perpendicular to $\alpha-\alpha_1$ the vector joining the centres of the spheres. This is always a real plane whether the spheres intersect or not. It is, in fact, what is called their *Radical Plane.*

221. Find the locus of a point the ratio of whose distances from two given points is constant.

Let the given points be O and A, the extremities of the vector α. Also let P be the required point in any of its positions, and $\overline{OP} = \rho$.

Then, at once, if n be the ratio of the lengths of the two lines,

$$T(\rho-\alpha) = nT\rho.$$

This gives $$\rho^2-2S\alpha\rho+\alpha^2 = n^2\rho^2,$$

or, by an easy transformation,

$$T\left(\rho - \frac{a}{1-n^2}\right) = T\left(\frac{na}{1-n^2}\right).$$

Thus the locus is a sphere whose radius is $T\left(\frac{na}{1-n^2}\right)$, and whose centre is at B, where $\overline{OB} = \frac{a}{1-n^2}$, a definite point in the line OA.

222. If in any line, OP, drawn from the origin to a given plane, OQ be taken such that $OQ.OP$ is constant, find the locus of Q.

Let $$Sa\rho = a$$

be the equation of the plane, ϖ a vector of the required surface. Then, by the conditions,

$$T\varpi\, T\rho = \text{constant} = b^2 \text{ (suppose)},$$

and $$U\varpi = U\rho.$$

From these $$\rho = \frac{b^2 U\varpi}{T\varpi} = -\frac{b^2\varpi}{\varpi^2}.$$

Substituting in the equation of the plane, we have

$$a\varpi^2 + b^2 Sa\varpi = 0,$$

which shows that the locus is a sphere, the origin being situated on it at the point farthest from the given plane.

223. Find the locus of points the sum of the squares of whose distances from a set of given points is a constant quantity. Find also the least value of this constant, and the corresponding locus.

Let the vectors from the origin to the given points be a_1, a_2, ...a_n, and to the sought point ρ, then

$$-c^2 = (\rho - a_1)^2 + (\rho - a_2)^2 + \ldots\ldots + (\rho - a_n)^2,$$
$$= n\rho^2 - 2S\rho\Sigma a + \Sigma(a^2).$$

Otherwise

$$\left(\rho - \frac{\Sigma a}{n}\right)^2 = -\frac{c^2 + \Sigma(a^2)}{n} + \frac{(\Sigma a)^2}{n^2},$$

the equation of a sphere the vector of whose centre is $\frac{\Sigma\alpha}{n}$, i.e. whose centre is the mean of the system of given points.

Suppose the origin to be placed at the mean point, the equation becomes
$$\rho^2 = -\frac{c^2 + \Sigma(\alpha^2)}{n}.$$
The right-hand side is negative, and therefore the equation denotes a real surface, if
$$c^2 > \Sigma T\alpha^2,$$
as might have been expected. When these quantities are equal, the locus becomes a point, viz. the origin.

224. If we differentiate the equation
$$T\rho = T\alpha$$
we get
$$S\rho d\rho = 0.$$
Hence (§ 137), ρ is *normal* to the surface at its extremity, a well-known property of the sphere.

If ϖ be any point in the plane which touches the sphere at the extremity of ρ, $\varpi - \rho$ is a line in the tangent plane, and therefore perpendicular to ρ. So that
$$S\rho(\varpi - \rho) = 0,$$
or
$$S\varpi\rho = -T\rho^2 = \alpha^2$$
is the equation of the tangent plane.

225. If this plane pass through a given point B, whose vector is β, we have
$$S\beta\rho = \alpha^2.$$
This is the equation of a plane, perpendicular to β, and cutting from it a portion whose length is
$$\frac{T\alpha^2}{T\beta}.$$
If this plane pass through a fixed point whose vector is γ we must have
$$S\beta\gamma = \alpha^2,$$

so that the locus of β is a plane. These results contain all the ordinary properties of poles and polars with regard to a sphere.

226. A line drawn parallel to γ, from the extremity of β, has the equation
$$\rho = \beta + x\gamma.$$
This meets the sphere
$$\rho^2 = \alpha^2$$
in points for which x has the values given by the equation
$$\beta^2 + 2xS\beta\gamma + x^2\gamma^2 = \alpha^2.$$
The values of x are imaginary, that is, there is no intersection, if
$$\alpha^2\gamma^2 + V^2\beta\gamma < 0.$$
The values are equal, or the line touches the sphere, if
$$\alpha^2\gamma^2 + V^2\beta\gamma = 0,$$
$$\text{or} \qquad S^2\beta\gamma = \gamma^2(\beta^2 - \alpha^2).$$
This is the equation of a cone similar and similarly situated to the cone of tangent-lines drawn to the sphere, but its vertex is at the centre. That the equation represents a cone is obvious from the fact that it is *homogeneous* in $T\gamma$, i. e. that it is independent of the length of the vector γ.

[It may be remarked that from the form of the above equation we see that, if x and x' be its roots, we have
$$(xT\gamma)(x'T\gamma) = \alpha^2 - \beta^2,$$
which is *Euclid*, III, 35, 36.]

227. Find the locus of the foot of the perpendicular let fall from a given point of a sphere on any tangent-plane.

Taking the centre as origin, the equation of any tangent-plane may be written
$$S\varpi\rho = \alpha^2.$$
The perpendicular must be parallel to ρ, so that, if we suppose it drawn from the extremity of α (which is a point on the sphere) we have as one value of ϖ
$$\varpi = \alpha + x\rho.$$

From these equations, with the help of that of the sphere

$$\rho^2 = a^2,$$

we must eliminate ρ and x.

We have by operating on the vector equation by $S.\varpi$

$$\begin{aligned}\varpi^2 &= S a\varpi + xS\varpi\rho \\ &= S a\varpi + xa^2.\end{aligned}$$

Hence $$\rho = \frac{\varpi - a}{x} = \frac{a^2(\varpi - a)}{\varpi^2 - S a\varpi}.$$

Taking the tensors, we have

$$(\varpi^2 - S a\varpi)^2 = a^2(\varpi - a)^2,$$

the required equation. It may be put in the form

$$S^2\varpi U(\varpi - a) = -a^2,$$

and the interpretation of this gives at once a characteristic property of the surface formed by the rotation of the *Cardioid* about its axis of symmetry.

228. We have seen that a sphere, referred to any point whatever as origin, has the equation

$$T(\rho - a) = T\beta.$$

Hence, to find the rectangle under the segments of a chord drawn through any point, we must put

$$\rho = x\gamma;$$

where γ is *any* unit-vector whatever. This gives

$$x^2\gamma^2 - 2xS a\gamma + a^2 = \beta^2,$$

and the product of the two values of x is

$$-\frac{\beta^2 - a^2}{\gamma^2} = -a^2 + \beta^2.$$

This is positive, or the vector-chords are drawn in the *same* direction, if $$T\beta < Ta,$$

i.e. if the origin is outside the sphere.

229. A, B are fixed points; and, O being the origin and P a point in space,

$$AP^2+BP^2 = OP^2;$$

find the locus of P, and explain the result when $\angle AOB$ is a right, or an obtuse, angle.

Let $\overline{OA} = a$, $\overline{OB} = \beta$, $\overline{OP} = \rho$, then

$$(\rho-a)^2+(\rho-\beta)^2 = \rho^2,$$

$$\text{or} \qquad \rho^2-2S(a+\beta)\rho = -(a^2+\beta^2),$$

$$\text{or} \qquad T\{\rho-(a+\beta)\} = \sqrt{(-2Sa\beta)}.$$

While $Sa\beta$ is negative, that is, while $\angle AOB$ is acute, the locus is a sphere whose centre has the vector $a+\beta$. If $Sa\beta = 0$, or $\angle AOB = \frac{\pi}{2}$, the locus is reduced to the point

$$\rho = a+\beta.$$

If $\angle AOB > \frac{\pi}{2}$ there is no point which satisfies the conditions.

230. Describe a sphere, with its centre in a given line, so as to pass through a given point and touch a given plane.

Let xa, where x is an undetermined scalar, be the vector of the centre, r the radius of the sphere, β the vector of the given point, and

$$S\gamma\rho = a$$

the equation of the given plane.

The vector perpendicular from the point xa on the given plane is (§ 208)

$$(a-xS\gamma a)\gamma^{-1}.$$

Hence, to determine x we have the equation

$$T.(a-xS\gamma a)\gamma^{-1} = T(xa-\beta) = r,$$

so that there are, in general, two solutions. It will be a good exercise for the student to find from this equation the condition that there may be no solution, or two coincident ones.

231. Describe a sphere whose centre is in a given line, and which passes through two given points.

Let the vector of the centre be $x\alpha$, as in last section, and let the vectors of the points be β and γ. Then, at once,

$$T(\gamma - x\alpha) = T(\beta - x\alpha) = r.$$

Here there is but *one* sphere, except in the particular case when we have

$$T\gamma = T\beta, \quad \text{and} \quad S\alpha\gamma = S\alpha\beta,$$

in which case there is an infinite number.

The student should carefully compare the results of this section and the last, so as to discover why in general two solutions are possible in the one case, and only one in the other.

232. A sphere touches each of two straight lines, which do not meet: find the locus of its centre.

We may take the origin at the middle point of the shortest distance (§ 203) between the given lines, and their equations will then be

$$\rho = \alpha + x\beta,$$
$$\rho = -\alpha + x_1\beta_1,$$

where we have, of course,

$$S\alpha\beta = 0, \qquad S\alpha\beta_1 = 0.$$

Let σ be the vector of the centre, ρ that of any point, of one of the spheres, and r its radius; its equation is

$$T(\rho - \sigma) = r.$$

Since the two given lines are tangents, the following equations in x and x_1 must have pairs of equal roots,

$$T(\alpha + x\beta - \sigma) = r,$$
$$T(-\alpha + x_1\beta_1 - \sigma) = r.$$

The equality of the roots in each gives us the conditions

$$S^2\beta\sigma = \beta^2((\alpha - \sigma)^2 + r^2),$$
$$S^2\beta_1\sigma = \beta_1^2((\alpha + \sigma)^2 + r^2).$$

Eliminating r we obtain

$$\beta^{-2}S^2\beta\sigma - \beta_1^{-2}S^2\beta_1\sigma = (\alpha - \sigma)^2 - (\alpha + \sigma)^2 = -4S\alpha\sigma,$$

which is the equation of the required locus.

[As we have not, so far, entered on the consideration of the quaternion form of the equations of the various surfaces of the second order, we may translate this into Cartesian cöordinates to find its meaning. If we take cöordinate axes of x, y, z respectively parallel to β, β_1, α, it becomes at once

$$(x+my)^2-(y+mx)^2 = pz,$$

where m and p are constants; and shows that the locus is a hyperbolic paraboloid. Such transformations, which are exceedingly simple in all cases, will be of frequent use to the student who is proficient in Cartesian geometry, in the early stages of his study of quaternions. As he acquires a practical knowledge of the new calculus, the need of such assistance will gradually cease to be felt.]

Simple as the above solution is, quaternions enable us to give one vastly simpler. For the problem may be thus stated—Find the locus of the point whose distances from two given lines are equal. And, with the above notation, the equality of the perpendiculars is expressed (§ 201) by

$$TV.(\alpha-\sigma)U\beta = TV.(\alpha+\sigma)U\beta_1,$$

which is easily seen to be equivalent to the equation obtained above.

233. Two spheres being given, show that spheres which cut them at given angles cut at right angles another fixed sphere.

If c be the distance between the centres of two spheres whose radii are a and b, the cosine of the angle of intersection is evidently

$$\frac{a^2+b^2-c^2}{2ab}.$$

Hence, if α, α_1, and ρ be the vectors of the centres, and a, a_1, r the radii, of the two fixed, and of one of the variable, spheres; A and A_1 the angles of intersection, we have

$$(\rho-\alpha)^2+a^2+r^2 = 2ar\cos A,$$

$$(\rho-\alpha_1)^2+a_1^2+r^2 = 2a_1r\cos A_1.$$

Eliminating the first power of r, we evidently must obtain a result such as

$$(\rho-\beta)^2+b^2+r^2=0,$$

where (by what precedes) β is the vector of the centre, and b the radius, of a fixed sphere

$$(\rho-\beta)^2+b^2=0,$$

which is cut at right angles by all the varying spheres. By effecting the elimination exactly we easily find b and β in terms of given quantities.

234. To inscribe in a given sphere a closed polygon, plane or gauche, whose sides shall be parallel respectively to each of a series of given vectors.

Let

$$T\rho=1$$

be the sphere, $\alpha, \beta, \gamma, \ldots\ldots, \eta, \theta$ the vectors, n in number, and let $\rho_1, \rho_2, \ldots\ldots \rho_n$, be the vector-radii drawn to the angles of the polygon.

Then

$$\rho_2-\rho_1=x_1\alpha, \text{ \&c., \&c.}$$

From this, by operating by $S.(\rho_2+\rho_1)$, we get

$$\rho_2^2-\rho_1^2=0=S\alpha\rho_2+S\alpha\rho_1.$$

Also

$$0=V\alpha\rho_2-V\alpha\rho_1.$$

Adding, we get

$$0=\alpha\rho_2+K\alpha\rho_1=\alpha\rho_2+\rho_1\alpha.$$

Hence

$$\rho_2=-\alpha^{-1}\rho_1\alpha.$$

[This might have been written down at once from the result of § 105.]

Similarly

$$\rho_3=-\beta^{-1}\rho_2\beta=\beta^{-1}\alpha^{-1}\rho_1\alpha\beta, \text{ \&c.}$$

Thus, finally, since the polygon is closed,

$$\rho_{n+1}=\rho_1=(-)^n\theta^{-1}\eta^{-1}\ldots\ldots\beta^{-1}\alpha^{-1}\rho_1\alpha\beta\ldots\ldots\eta\theta.$$

We may suppose the tensors of $\alpha, \beta\ldots\ldots\eta, \theta$ to be each unity.

Hence, if

$$a=\alpha\beta\ldots\ldots\eta\theta,$$

we have $$a^{-1} = \theta^{-1}\eta^{-1}\ldots\ldots\beta^{-1}a^{-1},$$
which is a known quaternion; and thus our condition becomes
$$\rho_1 = (-)^n a^{-1}\rho_1 a.$$
This divides itself into two cases, according as n is an even or an odd number.

If n be even, we have
$$a\rho_1 = \rho_1 a.$$
Removing the common part $\rho_1 Sa$, we have
$$V\rho_1 Va = 0.$$
This gives one determinate direction, $\pm Va$, for ρ_1; and shows that there are two, and only two, solutions.

If n be odd, we have
$$a\rho_1 = -\rho_1 a,$$
which requires that we have
$$Sa = 0.$$
Hence $$Sa\rho_1 = 0,$$
and therefore ρ_1 may be drawn to any point in the great circle of the unit-sphere whose poles are on the vector a.

235. To illustrate these results, let us take first the case of $n = 3$. Here we must have
$$S.a\beta\gamma = 0,$$
or the three given vectors must (as is obvious on other grounds) be parallel to one plane. Here $a\beta\gamma$, which lies in this plane, is (§ 106) the vector-tangent at the first corner of each of the inscribed triangles; and is obviously perpendicular to the vector drawn from the centre to that corner.

If $n = 4$, we have
$$\rho_1 \parallel V.a\beta\gamma\delta,$$
as might have been at once seen from § 106.

236. Hamilton has given (*Lectures*, p. 674) an ingenious and simple process by which the above investigation is rendered applicable to the more difficult problem in which each side of the inscribed polygon is to pass through a given point instead of being parallel to a given line. His process depends upon the integration of a linear equation in finite differences. By an immediate application of the linear and vector function of Chapter V, the above solutions may be at once extended to any central surface of the second order.

237. The equation of a cone of revolution, whose vertex is the origin, is easily found.

Suppose α, where $T\alpha = 1$, to be its axis, and e the cosine of its semi-vertical angle; then, if ρ be the vector of any point in the cone,

$$S\alpha U\rho = \mp e,$$

or

$$S^2\alpha\rho = -e^2\rho^2.$$

238. Change the origin to the point in the axis whose vector is $x\alpha$, and the equation becomes

$$(-x+S\alpha\varpi)^2 = -e^2(x\alpha+\varpi)^2.$$

Let the radius of the section of the cone made by

$$S\alpha\varpi = 0$$

retain a constant value b, while x changes; this necessitates

$$\frac{x}{\sqrt{b^2+x^2}} = e,$$

so that when x is infinite, e is unity. In this case the equation becomes

$$S^2\alpha\varpi + \varpi^2 + b^2 = 0,$$

which must therefore be the equation of a circular cylinder of radius b, whose axis is the vector α. To verify this we have only to notice that if ϖ be the vector of a point of such a cylinder we must (§ 201) have

$$TV\alpha\varpi = b,$$

which is the same equation as that above.

239. To find, generally, the equation of a cone which has circular sections:—

Take the origin as vertex, and let one of the circular sections be the intersection of the plane

$$S\alpha\rho = 1$$

with the sphere (passing through the origin)

$$\rho^2 = S\beta\rho.$$

These equations may be written thus,

$$S\alpha U\rho = \frac{1}{T\rho},$$

$$-T\rho = S\beta U\rho.$$

Hence, eliminating $T\rho$, we find the following equation which $U\rho$ must satisfy—

$$S\alpha U\rho\, S\beta U\rho = -1,$$

or $$\rho^2 - S\alpha\rho\, S\beta\rho = 0,$$

which is therefore the required equation of the cone.

As α and β are similarly involved, the mere *form* of this equation proves the existence of the subcontrary section discovered by Apollonius.

240. The equation just obtained may be written

$$S.U\alpha U\rho S.U\beta U\rho = -\frac{1}{T.\alpha\beta},$$

or, since α and β are perpendicular to the cyclic arcs (§ 59*),

$$\sin p \sin p' = \text{constant},$$

where p and p' are arcs drawn from any point of a spherical conic perpendicular to the cyclic arcs. This is a well-known property of such curves.

241. If we cut the cyclic cone by any plane passing through the origin, as $$S\gamma\rho = 0,$$

then $V\alpha\gamma$ and $V\beta\gamma$ are the traces on the cyclic planes, so that

$$\rho = xUV\alpha\gamma + yUV\beta\gamma \quad (\S\ 29).$$

Substitute in the equation of the cone, and we get

$$-x^2-y^2+Pxy = 0,$$

where P is a known scalar. Hence the values of x and y are the same pair of numbers. This is a very elementary proof of the proposition in § 59*, that $\widehat{PL} = \widehat{MQ}$ (in the last figure of that section).

242. When x and y are *equal*, the transversal arc becomes a tangent to the spherical conic, and is evidently bisected at the point of contact. Here we have

$$P = 2 = 2S.UVa\gamma UV\beta\gamma + \frac{(S.a\beta\gamma)^2}{T.Va\gamma V\beta\gamma}.$$

This is the equation of the cone whose sides are perpendiculars (through the origin) to the planes which touch the cyclic cone.

243. It may be well to observe that the property of the Stereographic projection of the sphere, viz. that the projection of a circle is a circle, is an immediate consequence of the above form of the equation of a cyclic cone.

244. That § 239 gives the most general form of the equation of a cone of the second order, when the vertex is taken as origin, follows from the early results of next Chapter. For it is shown in § 249 that the equation of a cone of the second order can always be put in the form

$$2\Sigma.Sa\rho S\beta\rho + A\rho^2 = 0.$$

This may be written $\quad S\rho\phi\rho = 0,$

where ϕ is the self-conjugate linear and vector function

$$\phi\rho = \Sigma V.a\rho\beta + (A + \Sigma Sa\beta)\rho.$$

By § 168 this may be transformed to

$$\phi\rho = p\rho + V.\lambda\rho\mu,$$

and the general equation of the cone becomes

$$(p - S\lambda\mu)\rho^2 + 2S\lambda\rho S\mu\rho = 0,$$

which is the form obtained in § 239.

245. Taking the form

$$S\rho\phi\rho = 0$$

as the simplest, we find by differentiation

$$Sd\rho\phi\rho + S\rho d\phi\rho = 0,$$

or $$2Sd\rho\phi\rho = 0.$$

Hence $\phi\rho$ is perpendicular to the tangent-plane at the extremity of ρ. The equation of this plane is therefore (ϖ being the vector of any point in it)

$$S\phi\rho(\varpi - \rho) = 0,$$

or, by the equation of the cone,

$$S\varpi\phi\rho = 0.$$

246. The equation of the cone of normals to the tangent-planes of the given cone can be easily formed from that of the cone itself. For we may write it in the form

$$S(\phi^{-1}\phi\rho)\phi\rho = 0,$$

and if we put $\phi\rho = \sigma$, a vector of the new cone, the equation becomes
$$S\sigma\phi^{-1}\sigma = 0.$$

Numerous curious properties of these connected cones, and of the corresponding spherical conics, follow at once from these equations. But we must leave them to the reader.

247. As a final example, let us find the equation of a cyclic cone when *five* of its vector-sides are given—i. e. find the cone of the second order whose vertex is the origin, and on whose surface lie the vectors $\alpha, \beta, \gamma, \delta, \epsilon$.

If we write

$$0 = S.V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha), \text{ .... } (1)$$

we have the equation of a *cone* whose vertex is the origin—for the equation is not altered by putting $x\rho$ for ρ. Also it is the equation of a cone of the second degree, since ρ occurs only twice. Moreover the vectors $\alpha, \beta, \gamma, \delta, \epsilon$ are sides of the cone, because if any one of them be put for ρ the equation is satisfied. Thus if we put β for ρ the equation becomes

$$0 = S.V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\beta)V(V\gamma\delta V\beta\alpha)$$
$$= S.V(V\alpha\beta V\delta\epsilon)\{V\beta\alpha S.V\gamma\delta V\beta\gamma V\epsilon\beta - V\gamma\delta S.V\beta\alpha V\beta\gamma V\epsilon\beta\}.$$

The first term vanishes because

$$S.V(V\alpha\beta V\delta\epsilon)V\beta\alpha = 0,$$

and the second because

$$S.V\beta\alpha V\beta\gamma V\epsilon\beta = 0,$$

since the three vectors $V\beta\alpha$, $V\beta\gamma$, $V\epsilon\beta$, being each at right angles to β, must be in one plane.

As is remarked by Hamilton, this is a very simple proof of Pascal's Theorem—for (1) is the condition that the intersections of the planes of α, β and δ, ϵ; β, γ and ϵ, ρ; γ, δ and ρ, α; shall lie in one plane; or, making the statement for any plane section of the cone, the points of intersection of the three pairs of opposite sides, of a hexagon inscribed in a conic, lie in one straight line.

EXAMPLES TO CHAPTER VII.

1. On the vector of a point P in the plane

$$S\alpha\rho = 1$$

a point Q is taken, such that $QO.OP$ is constant; find the equation of the locus of Q.

2. What spheres cut the loci of P and Q in (1) so that both lines of intersection lie on a cone whose vertex is O?

3. A sphere touches a fixed plane, and cuts a fixed sphere. If the point of contact with the plane be given, the plane of the intersection of the spheres contains a fixed line.

Find the locus of the centre of the variable sphere, if the plane of its intersection with the fixed sphere passes through a given point.

4. Find the radii of the spheres which touch, simultaneously, the four given planes

$$S\alpha\rho = 0, \qquad S\beta\rho = 0, \qquad S\gamma\rho = 0, \qquad S\delta\rho = 1.$$

[What is the volume of the tetrahedron enclosed by these planes?]

5. If a moveable line, passing through the origin, make with any number of fixed lines angles θ, θ_1, θ_2, &c., such that

$$a\cos.\theta + a_1\cos.\theta_1 + \ldots\ldots = \text{constant},$$

where a, a_1, ... are constant scalars, the line describes a right cone.

6. Determine the conditions that

$$S\rho\phi\rho = 0$$

may represent a *right* cone.

7. What property of a cone (or of a spherical conic) is given directly by the following form of its equation,

$$S.\iota\rho\kappa\rho = 0\ ?$$

8. What are the conditions that the surfaces represented by

$$S\rho\phi\rho = 0, \qquad \text{and} \qquad S.\iota\rho\kappa\rho = 0,$$

may degenerate into pairs of planes?

9. Find the locus of the vertices of all right cones which have a common ellipse as base.

10. Two right circular cones have their axes parallel, show that the orthogonal projection of their curve of intersection on the plane containing their axes is a parabola.

11. Two spheres being given in magnitude and position, every sphere which intersects them in given angles will touch two other fixed spheres and cut another at right angles.

12. If a sphere be placed on a table, the breadth of the elliptic shadow formed by rays diverging from a fixed point is independent of the position of the sphere.

13. Form the equation of the cylinder which has a given circular section, and a given axis. Find the direction of the normal to the subcontrary section.

14. Given the base of a spherical triangle, and the product of the cosines of the sides, the locus of the vertex is a spherical conic, the poles of whose cyclic arcs are the extremities of the given base.

15. (Hamilton, *Bishop Law's Premium Ex.*, 1858.)

(*a*.) What property of a sphero-conic is most immediately indicated by the equation

$$S\frac{\rho}{\alpha}\,S\frac{\beta}{\rho} = 1\,?$$

(*b*.) The equation

$$(V\lambda\rho)^2 + (S\mu\rho)^2 = 0$$

also represents a cone of the second order; λ is a focal line, and μ is perpendicular to the director-plane corresponding.

(*c*.) What property of a sphero-conic does the equation most immediately indicate?

16. Show that the areas of all triangles, bounded by a tangent to a spherical conic and the cyclic arcs, are equal.

17. Show that the locus of a point, the sum of whose arcual distances from two given points on a sphere is constant, is a spherical conic.

18. If two tangent planes be drawn to a cyclic cone, the four lines in which they intersect the cyclic planes are sides of a right cone.

19. Find the equation of the cone whose sides are the intersections of pairs of perpendicular tangent planes to a given cyclic cone.

20. Find the condition that five given points may lie on a sphere.

21. What is the surface denoted by the equation

$$\rho^2 = x\alpha^2 + y\beta^2 + z\gamma^2,$$

where

$$\rho = x\alpha + y\beta + z\gamma,$$

α, β, γ being given vectors, and x, y, z variable scalars?

Express the equation of the surface in terms of $\rho, \alpha, \beta, \gamma$ alone.

22. Find the equation of the cone whose sides bisect the angles between a fixed line and any line, in a given plane, which meets the fixed line.

What property of a spherical conic is most directly given by this result?

CHAPTER VIII.

SURFACES OF THE SECOND ORDER.

248. THE general scalar equation of the second order in a vector ρ must evidently contain a term independent of ρ, terms of the form $S.a\rho b$ involving ρ to the first degree, and others of the form $S.a\rho b\rho c$ involving ρ to the second degree, a, b, c, &c. being constant quaternions. Now the term $S.a\rho b$ may be written

$$S.(Sa+Va)\rho(Sb+Vb),$$

or $$SaS\rho Vb+SbS\rho Va+S.\rho VbVa,$$

each of which may evidently be put in the form $S\gamma\rho$, where γ is a known vector.

Similarly the term $S.a\rho b\rho c$ may be reduced to a set of terms, each of which has one of the forms

$$A\rho^2, \quad (S\alpha\rho)^2, \quad S\alpha\rho S\beta\rho,$$

the second being merely a particular case of the third. Thus (the numerical factors 2 being introduced for convenience) we may write the general scalar equation of the second degree as follows:—

$$2\Sigma.S\alpha\rho S\beta\rho+A\rho^2+2S\gamma\rho=C. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

249. Change the origin to D where $OD=\delta$, then ρ becomes $\rho+\delta$, and the equation takes the form

$$2\Sigma.S\alpha\rho S\beta\rho+A\rho^2+2\Sigma(S\alpha\rho S\beta\delta+S\beta\rho S\alpha\delta)+2AS\delta\rho+2S\gamma\rho$$
$$+2\Sigma.S\alpha\delta S\beta\delta+A\delta^2+2S\gamma\delta-C=0;$$

from which the first power of ρ disappears, that is *the surface is referred to its centre*, if

$$\Sigma(\alpha S\beta\delta+\beta S\alpha\delta)+A\delta+\gamma=0, \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

a vector equation of the first degree, which in general gives a single definite value for δ, by the processes of Chapter V. [It would lead us beyond the limits of an elementary treatise to consider the special cases in which (2) represents a line, or a plane, any point of which is a centre of the surface. The processes to be employed in such special cases have been amply illustrated in the Chapter referred to.]

With this value of δ, and putting

$$D = C - 2S\gamma\delta - A\delta^2 - 2\Sigma . S\alpha\delta S\beta\delta,$$

the equation becomes

$$2\Sigma . S\alpha\rho S\beta\rho + A\rho^2 = D.$$

If $D = 0$, the surface is conical (a case treated in last Chapter); if not, it is an ellipsoid or hyperboloid. Unless expressly stated not to be, the surface will, when D is not zero, be considered an ellipsoid. By this we avoid for the time some rather delicate considerations.

By dividing by D, and thus altering only the tensors of the constants, we see that the equation of central surfaces of the second order, referred to the centre, is (excluding cones)

$$2\Sigma(S\alpha\rho S\beta\rho) + g\rho^2 = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

250. Differentiating, we obtain

$$2\Sigma\{S\alpha d\rho S\beta\rho + S\alpha\rho S\beta d\rho\} + 2gS\rho d\rho = 0,$$

or $$S.d\rho\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 0,$$

and therefore, by § 137, the tangent plane is

$$S(\varpi - \rho)\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 0,$$

i. e. $$S.\varpi\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 1, \text{ by (3).}$$

Hence if $$\nu = \Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

the tangent plane is $S\nu\varpi = 1$,

and the surface itself is $S\nu\rho = 1$.

And, as ν^{-1} is evidently the vector-perpendicular from the origin on the tangent plane, ν is called the *vector of proximity*.

251. Hamilton uses for ν, which is obviously a linear and vector function of ρ, the notation $\phi\rho$, ϕ expressing a functional operation, as in Chapter V. But, for the sake of clearness, we will go over part of the ground again, especially for the benefit of students who have mastered only the more elementary parts of that Chapter.

We have, then,

$$\phi\rho = \Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho.$$

With this definition of ϕ, it is easy to see that

(*a.*) $\phi(\rho+\sigma) = \phi\rho + \phi\sigma$, &c., for *any* two or *more* vectors.

(*b.*) $\phi(x\rho) = x\phi\rho$, a particular case of (*a*), x being a scalar.

(*c.*) $d\phi\rho = \phi(d\rho)$.

(*d.*) $S\sigma\phi\rho = \Sigma(S\alpha\sigma S\beta\rho + S\beta\sigma S\alpha\rho) + gS\rho\sigma = S\rho\phi\sigma$,

or ϕ is, in this case, self-conjugate.

This last property is of great importance.

252. Thus the general equation of central surfaces of the second degree (excluding cones) may now be written

$$S\rho\phi\rho = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Differentiating, $\quad S d\rho\phi\rho + S\rho d\phi\rho = 0,$

which, by applying (*c.*) and then (*d.*) to the last term on the left, gives

$$2S\phi\rho d\rho = 0,$$

and therefore, as in § 250, though now much more simply, the tangent plane at the extremity of ρ is

$$S(\varpi-\rho)\phi\rho = 0,$$

or $\quad S\varpi\phi\rho = S\rho\phi\rho = 1.$

If this pass through A ($\overline{OA} = \alpha$), we have

$$S\alpha\phi\rho = 1,$$

or, by (*d.*), $\quad S\rho\phi\alpha = 1,$

for all possible points of contact.

This is therefore the equation of the plane of contact of tangent planes drawn from A.

253. To find the enveloping cone whose vertex is A, notice that

$$(S\rho\phi\rho - 1) + p(S\rho\phi\alpha - 1)^2 = 0,$$

where p is any scalar, is the equation of a surface of the second order *touching* the ellipsoïd along its intersection with the plane. If this pass through A we have

$$(S\alpha\phi\alpha - 1) + p(S\alpha\phi\alpha - 1)^2 = 0,$$

and p is found. Then our equation becomes

$$(S\rho\phi\rho - 1)(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha - 1)^2 = 0, \quad \dots\dots\dots \quad (1)$$

which is the cone required. To assure ourselves of this, transfer the origin to A, by putting $\rho + \alpha$ for ρ. The result is, using (*a.*) and (*d.*),

$$(S\rho\phi\rho + 2S\rho\phi\alpha + S\alpha\phi\alpha - 1)(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha + S\alpha\phi\alpha - 1)^2 = 0,$$

or
$$S\rho\phi\rho(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha)^2 = 0,$$

which is homogeneous in $T\rho^2$, and is therefore the equation of a cone.

Suppose A infinitely distant, then we may put in (1) $x\alpha$ for α, where x is infinitely great, and, omitting all but the higher terms, the equation of the cylinder formed by tangent lines parallel to α is

$$(S\rho\phi\rho - 1)S\alpha\phi\alpha - (S\rho\phi\alpha)^2 = 0.$$

254. To study the nature of the surface more closely, let us *find the locus of the middle points of a system of parallel chords.*

Let them be parallel to α, then, if ϖ be the vector of the middle point of one of them, $\varpi + x\alpha$ and $\varpi - x\alpha$ are simultaneous values of ρ which ought to satisfy (1) of § 252.

That is
$$S.(\varpi \pm x\alpha)\phi(\varpi \pm x\alpha) = 1.$$

Hence, by (*a.*) and (*d.*), as before,

$$S\varpi\phi\varpi + x^2 S\alpha\phi\alpha = 1,$$

$$S\varpi\phi\alpha = 0. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

The latter equation shows that the locus of the extremity of ϖ, the middle point of a chord parallel to α, is a plane through the centre, whose normal is $\phi\alpha$; that is, a plane parallel to the tangent plane at the point where OA cuts the surface. And (*d.*) shows that this relation is reciprocal—so that if β be *any* value of ϖ, i.e. be any vector in the plane (1), α will be a vector in a diametral plane which bisects all chords parallel to β. The equations of these planes are

$$S\varpi\phi\alpha = 0,$$
$$S\varpi\phi\beta = 0,$$

so that if $V.\phi\alpha\phi\beta = \gamma$ (suppose) is their line of intersection, we have

$$\left.\begin{aligned} S\gamma\phi\alpha &= 0 = S\alpha\phi\gamma \\ S\gamma\phi\beta &= 0 = S\beta\phi\gamma \\ S\beta\phi\alpha &= 0 = S\alpha\phi\beta \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

and (1) gives [the last of these].

Hence there is *an infinite number of sets of three vectors α, β, γ, such that all chords parallel to any one are bisected by the diametral plane containing the other two.*

255. It is evident from § 23 that any vector may be expressed as a linear function of any three others not in the same plane, let then

$$\rho = x\alpha + y\beta + z\gamma,$$

where, by last section,

$$S\alpha\phi\beta = S\beta\phi\alpha = 0,$$
$$S\alpha\phi\gamma = S\gamma\phi\alpha = 0,$$
$$S\beta\phi\gamma = S\gamma\phi\beta = 0.$$

And let

$$\left.\begin{aligned} S\alpha\phi\alpha &= 1 \\ S\beta\phi\beta &= 1 \\ S\gamma\phi\gamma &= 1 \end{aligned}\right\},$$

so that α, β, and γ are vector conjugate semi-diameters of the surface we are engaged on.

Substituting the above value of ρ in the equation of the surface, and attending to the equations in α, β, γ and to (*a.*), (*b.*), and (*d.*), we have

$$S\rho\phi\rho = S(x\alpha + y\beta + z\gamma)\,\phi\,(x\alpha + y\beta + z\gamma),$$
$$= x^2 + y^2 + z^2 = 1.$$

To transform this equation to Cartesian coördinates, we notice that x is the ratio which the projection of ρ on α bears to α itself, &c. If therefore we take the conjugate diameters as axes of ξ, η, ζ, and their lengths as a, b, c, the above equation becomes at once

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1,$$

the ordinary equation of the ellipsoid referred to conjugate diameters.

256. If we write $-\psi^2$ instead of ϕ, these equations assume an interesting form. We take for granted, what we shall afterwards prove, that this halving or extracting the root of the vector function is lawful, and that the new linear and vector function has the same properties (*a.*), (*b.*), (*c.*), (*d.*) (§ 251) as the old. The equation of the surface now becomes

$$S\rho\psi^2\rho = -1,$$

or $$S\psi\rho\psi\rho = -1,$$

or, finally, $$T\psi\rho = 1.$$

If we compare this with the equation of the unit-sphere

$$T\rho = 1,$$

we see at once the analogy between the two surfaces. *The sphere can be changed into the ellipsoid, or vice versâ, by a linear deformation of each vector, the operator being the function* ψ *or its inverse.* See the Chapter on Physical Applications.

257. Equations (2) § 254 now become

$$S\alpha\psi^2\beta = 0 = S\psi\alpha\psi\beta, \text{ \&c.}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

so that $\psi\alpha$, $\psi\beta$, $\psi\gamma$, *the vectors of the unit-sphere which cor-*

respond to semi-conjugate diameters of the ellipsoid, form a rectangular system.

We may remark here, that, as the equation of the ellipsoid referred to its principal axes is a case of § 255, we may now suppose $i, j,$ and k to have these directions, and the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, which, in quaternions, is

$$S\rho\phi\rho = \frac{(Si\rho)^2}{a^2} + \frac{(Sj\rho)^2}{b^2} + \frac{(Sk\rho)^2}{c^2} = 1.$$

We here tacitly assume the existence of such axes, but in all cases, by the help of Hamilton's method, developed in Chapter V, we at once arrive at the cubic equation which gives them.

It is evident from the last-written equation that

$$\phi\rho = + \frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2},$$

and $$\psi\rho = -\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right),$$

which latter may be easily proved by showing that

$$\psi^2\rho = -\phi\rho.$$

And this expression enables us to verify the assertion of last section about the properties of ψ.

As $Si\rho = -x$, &c., x, y, z being the Cartesian coördinates referred to the principal axes, we have now the means of at once transforming any quaternion result connected with the ellipsoid into the ordinary one.

258. Before proceeding to other forms of the equation of the ellipsoid, we may use those already given in solving a few problems.

Find the locus of a point when the perpendicular from the centre on its polar plane is of constant length.

If ϖ be the vector of the point, the polar plane is

$$S\rho\phi\varpi = 1,$$

and the length of the perpendicular from O is $\frac{1}{T\phi\varpi}$ (§ 208).

Hence the required locus is

$$T\phi\varpi = C,$$

or

$$S\varpi\phi^2\varpi = -C^2,$$

a concentric ellipsoid, with its axes in the same direction as those of the first. By § 257 its Cartesian equation is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = C^2.$$

259. *Find the locus of a point whose distance from a given point is always in a given ratio to its distance from a given line.*

Let $\rho = x\beta$ be the given line, and A ($\overline{OA} = \alpha$) the given point, and let $S\alpha\beta = 0$. Then for any one of the required points

$$T(\rho-\alpha) = e\,TV\beta\rho,$$

a surface of the second order, which may be written

$$\rho^2 - 2S\alpha\rho + \alpha^2 = e^2(S^2\beta\rho - \beta^2\rho^2).$$

Let the centre be at δ, and make it the origin, then

$$\rho^2 + 2S\rho(\delta-\alpha) + (\delta-\alpha)^2 = e^2\{S^2.\beta(\rho+\delta) - \beta^2(\rho+\delta)^2\},$$

and, that the first power of ρ may disappear,

$$(\delta-\alpha) = e^2(\beta S\beta\delta - \beta^2\delta),$$

a linear equation for δ. To solve it, note that $S\alpha\beta = 0$, operate by $S.\beta$ and we get

$$(1 - e^2\beta^2 + e^2\beta^2)S\beta\delta = S\beta\delta = 0.$$

Hence

$$\delta - \alpha = -e^2\beta^2\delta,$$

or

$$\delta = \frac{\alpha}{1+e^2\beta^2}.$$

Referred to this point as origin the equation becomes

$$(1+e^2\beta^2)\rho^2 - e^2S^2\beta\rho + \frac{e^2\beta^2\alpha^2}{1+e^2\beta^2} = 0,$$

which shows that it belongs to a surface of revolution whose axis is parallel to β, as its intersection with a plane $S\beta\rho = a$, perpendicular to that axis, lies also on the sphere

$$\rho^2 = \frac{e^2a^2}{1+e^2\beta^2} - \frac{e^2\beta^2\alpha^2}{(1+e^2\beta^2)^2}.$$

260. *A sphere, passing through the centre of an ellipsoid, is cut by a series of spheres whose centres are on the ellipsoid and which pass through the centre thereof; find the envelop of the planes of intersection.*

Let $(\rho-\alpha)^2 = \alpha^2$ be the first sphere, i.e.

$$\rho^2 - 2S\alpha\rho = 0.$$

One of the others is

$$\rho^2 - 2S\varpi\rho = 0,$$

where $$S\varpi\phi\varpi = 1.$$

The plane of intersection is

$$S(\varpi-\alpha)\rho = 0.$$

Hence, for the envelop, (see next Chapter,)

$$\left.\begin{aligned} S\varpi'\phi\varpi &= 0, \\ S\varpi'\rho &= 0, \end{aligned}\right\} \text{ where } \varpi' = d\varpi,$$

or $$\phi\varpi = x\rho, \quad \{Vx = 0\},$$

i.e. $$\varpi = x\phi^{-1}\rho.$$

Hence $$\left.\begin{aligned} x^2 S\rho\phi^{-1}\rho &= 1, \\ xS\rho\phi^{-1}\rho &= S\alpha\rho, \end{aligned}\right\}$$

and, eliminating x,

$$S\rho\phi^{-1}\rho = (S\alpha\rho)^2,$$

a cone of the second order.

261. *From a point in the outer of two concentric ellipsoids a tangent cone is drawn to the inner, find the envelop of the plane of contact.*

If $S\varpi\phi\varpi = 1$ be the outer, and $S\rho\psi\rho = 1$ be the inner, ϕ and ψ being any two self-conjugate linear and vector functions, the plane of contact is

$$S\varpi\psi\rho = 1.$$

Hence, for the envelop, $$\left.\begin{aligned} S\varpi'\psi\rho &= 0, \\ S\varpi'\phi\varpi &= 0, \end{aligned}\right\}$$

therefore $$\phi\varpi = x\psi\rho,$$

or $$\varpi = x\phi^{-1}\psi\rho.$$

This gives
$$\left.\begin{aligned} xS.\psi\rho\phi^{-1}\psi\rho &= 1, \\ \text{and}\quad x^2S.\psi\rho\phi^{-1}\psi\rho &= 1, \end{aligned}\right\}$$
and therefore, eliminating x,
$$S.\psi\rho\phi^{-1}\psi\rho = 1,$$
or
$$S.\rho\psi\phi^{-1}\psi\rho = 1,$$
another concentric ellipsoid, as $\psi\phi^{-1}\psi$ is a linear and vector function $= \chi$ suppose; so that the equation may be written
$$S\rho\chi\rho = 1.$$

262. *Find the locus of intersection of tangent planes at the extremities of conjugate diameters.*

If α, β, γ be the vector semi-diameters, the planes are
$$\left.\begin{aligned} S\varpi\psi^2\alpha &= -1, \\ S\varpi\psi^2\beta &= -1, \\ S\varpi\psi^2\gamma &= -1, \end{aligned}\right\}$$
with the conditions § 257.

Hence.
$$-\psi\varpi S.\psi\alpha\psi\beta\psi\gamma = \psi\varpi = \psi\alpha+\psi\beta+\psi\gamma, \text{ by § 92,}$$
therefore
$$T\psi\varpi = \sqrt{3},$$
since $\psi\alpha$, $\psi\beta$, $\phi\gamma$ form a rectangular system of unit-vectors.

This may also evidently be written
$$S\varpi\psi^2\varpi = -3,$$
showing that the locus is similar and similarly situated to the given ellipsoid, but larger in the ratio $\sqrt{3} : 1$.

263. *Find the locus of the intersection of three spheres whose diameters are semi-conjugate diameters of an ellipsoid.*

If α be one of the semi-conjugate diameters
$$S\alpha\psi^2\alpha = -1.$$

And the corresponding sphere is
$$\rho^2 - S\alpha\rho = 0,$$
or
$$\rho^2 - S\psi\alpha\psi^{-1}\rho = 0,$$

with similar equations in β and γ. Hence, by § 92,

$$\psi^{-1}\rho\, S.\psi\alpha\psi\beta\psi\gamma = -\psi^{-1}\rho = \rho^2(\psi\alpha + \psi\beta + \psi\gamma),$$

and, taking tensors,

$$T\psi^{-1}\rho = \sqrt{3}\, T\rho^2,$$

or

$$T\psi^{-1}\rho^{-1} = \sqrt{3},$$

or, finally,

$$S\rho\psi^{-2}\rho = -3\rho^4.$$

This is Fresnel's *Surface of Elasticity* in the Undulatory Theory.

264. Before going farther we may prove some useful properties of the function ϕ in the form we are at present using—viz.

$$\phi\rho = \frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}.$$

We have

$$\rho = -iSi\rho - jSj\rho - kSk\rho,$$

and it is evident that

$$\phi i = -\frac{i}{a^2}, \quad \phi j = -\frac{j}{b^2}, \quad \phi k = -\frac{k}{c^2}.$$

Hence

$$\phi^2\rho = -\frac{iSi\rho}{a^4} - \frac{jSj\rho}{b^4} - \frac{kSk\rho}{c^4}.$$

Also

$$\phi^{-1}\rho = a^2 iSi\rho + b^2 jSj\rho + c^2 kSk\rho,$$

and so on.

Again, if α, β, γ be *any* rectangular unit-vectors

$$S\alpha\phi\alpha = \frac{(Si\alpha)^2}{a^2} + \frac{(Sj\alpha)^2}{b^2} + \frac{(Sk\alpha)^2}{c^2},$$

$$\&c. = \&c.$$

But as

$$(Si\rho)^2 + (Sj\rho)^2 + (Sk\rho)^2 = -\rho^2,$$

we have

$$S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Again,

$$S.\phi\alpha\phi\beta\phi\gamma = S.\left(\frac{iSi\alpha}{a^2} + \ldots\right)\left(\frac{iSi\beta}{a^2} + \ldots\right)\left(\frac{iSi\gamma}{a^2} + \ldots\right)$$

$$= -\begin{vmatrix} \frac{Si\alpha}{a^2}, & \frac{Sj\alpha}{b^2}, & \frac{Sk\alpha}{c^2} \\ \frac{Si\beta}{a^2}, & \frac{Sj\beta}{b^2}, & \frac{Sk\beta}{c^2} \\ \frac{Si\gamma}{a^2}, & \frac{Sj\gamma}{b^2}, & \frac{Sk\gamma}{c^2} \end{vmatrix} = \frac{-1}{a^2b^2c^2}\begin{vmatrix} Si\alpha, & Sj\alpha, & Sk\alpha \\ Si\beta, & Sj\beta, & Sk\beta \\ Si\gamma, & Sj\gamma, & Sk\gamma \end{vmatrix} = \frac{1}{a^2b^2c^2}.$$

And so on. These elementary investigations are given here for the benefit of those who have not read Chapter V. The student may easily obtain all such results in a far more simple manner by means of the formulæ of that Chapter.

265. *Find the locus of intersection of a rectangular system of three tangents to an ellipsoid.*

If ϖ be the vector of the point of intersection, α, β, γ the tangents, then, since $\varpi + x\alpha$ should give equal values of x when substituted in the equation of the surface, giving

$$S(\varpi + x\alpha)\phi(\varpi + x\alpha) = 1,$$

or
$$x^2 S\alpha\phi\alpha + 2xS\varpi\phi\alpha + (S\varpi\phi\varpi - 1) = 0,$$

we have
$$(S\varpi\phi\alpha)^2 = S\alpha\phi\alpha(S\varpi\phi\varpi - 1).$$

Adding this to the two similar equations in β and γ

$$(S\alpha\phi\varpi)^2 + (S\beta\phi\varpi)^2 + (S\gamma\phi\varpi)^2 = (S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma)(S\varpi\phi\varpi - 1),$$

or
$$-(\phi\varpi)^2 = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)(S\varpi\phi\varpi - 1),$$

or
$$S.\varpi\left\{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\phi + \phi^2\right\}\varpi = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2},$$

an ellipsoid concentric with the first.

266. *If a rectangular system of chords be drawn through any point within an ellipsoid, the sum of the reciprocals of the rectangles under the segments into which they are divided is constant.*

With the notation of the solution of the preceding problem, ϖ giving the intersection of the vectors, it is evident that the product of the values of x is one of the rectangles in question taken negatively.

Hence the required sum is

$$-\frac{\Sigma S\alpha\phi\alpha}{S\varpi\phi\varpi - 1} = -\frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}{S\varpi\phi\varpi - 1}.$$

This evidently depends on $S\varpi\phi\varpi$ only and not on the particular directions of α, β, γ: and is therefore unaltered if ϖ be the vector of any point of an ellipsoid similar, and similarly situated, to the given one. [The expression is interpretable even if the point be exterior to the ellipsoid.]

267. *Show that if any rectangular system of three vectors be drawn from a point of an ellipsoid, the plane containing their other extremities passes through a fixed point. Find the locus of the latter point as the former varies.*

With the same notation as before, we have

$$S\varpi\phi\varpi = 1,$$

and
$$S(\varpi + x\alpha)\phi(\varpi + x\alpha) = 1;$$

therefore
$$x = -\frac{2S\alpha\phi\varpi}{S\alpha\phi\alpha}.$$

Hence the required plane passes through the extremity of

$$\varpi - 2\alpha\frac{S\alpha\phi\varpi}{S\alpha\phi\alpha},$$

and those of two other vectors similarly determined. It therefore passes through the point whose vector is

$$\theta = \varpi - 2\frac{\alpha S\alpha\phi\varpi + \beta S\beta\phi\varpi + \gamma S\gamma\phi\varpi}{S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma},$$

or
$$\theta = \varpi + \frac{2\phi\varpi}{m_2} \quad (\S\ 173).$$

Thus the first part of the proposition is proved.

But we have also

$$\varpi = \frac{m_2}{2}\left(\phi + \frac{m_2}{2}\right)^{-1}\theta,$$

whence by the equation of the ellipsoid

$$\frac{m_2^2}{4}S.\theta\left(\phi + \frac{m_2}{2}\right)^{-1}\phi\left(\phi + \frac{m_2}{2}\right)^{-1}\theta = 1,$$

the equation of a concentric ellipsoid.

268. *Find the directions of the three vectors which are parallel to a set of conjugate diameters in each of two central surfaces of the second degree.*

Transferring the centres of both to the origin, let their equations be

$$\left.\begin{array}{rl} & S\rho\phi\rho = 1 \text{ or } 0, \\ \text{and} & S\rho\psi\rho = 1 \text{ or } 0. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

If α, β, γ be vectors in the required directions, we must have (§ 254)

$$\left.\begin{array}{ll} S\alpha\phi\beta = 0, & S\alpha\psi\beta = 0, \\ S\beta\phi\gamma = 0, & S\beta\psi\gamma = 0, \\ S\gamma\phi\alpha = 0, & S\gamma\psi\alpha = 0. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

From these equations

$$\phi\alpha \,\|\, V\beta\gamma \,\|\, \psi\alpha, \text{ \&c.}$$

Hence the three required directions are the roots of

$$V.\phi\rho\psi\rho = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

This is evident on other grounds, for it means that *if one of the surfaces expand or contract uniformly till it meets the other, it will touch it successively at points on the three sought vectors.*

We may put (3) in either of the following forms—

$$\left.\begin{array}{rl} & V.\rho\phi^{-1}\psi\rho = 0, \\ \text{or} & V.\rho\psi^{-1}\phi\rho = 0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

and, as ϕ and ψ are given functions, we find the solutions by the processes of Chapter V.

[*Note.* As $\phi^{-1}\psi$ and $\psi^{-1}\phi$ are not, in general, self-conjugate functions, equations (4) do not signify that α, β, γ are vectors parallel to the principal axes of the surfaces

$$S.\rho\phi^{-1}\psi\rho = 1, \qquad S.\rho\psi^{-1}\phi\rho = 1.]$$

269. *Find the equation of the ellipsoid of which three conjugate semi-diameters are given.*

Let the vector semi-diameters be α, β, γ, and let

$$S\rho\phi\rho = 1$$

be the equation of the ellipsoid. Then (§ 255) we have

$$\begin{aligned} S\alpha\phi\alpha &= 1, & S\alpha\phi\beta &= 0, \\ S\beta\phi\beta &= 1, & S\beta\phi\gamma &= 0, \\ S\gamma\phi\gamma &= 1, & S\gamma\phi\alpha &= 0\,; \end{aligned}$$

the six scalar conditions requisite (§ 139) for the determination of the linear and vector function ϕ.

They give
$$\alpha \parallel V\phi\beta\phi\gamma,$$
or
$$x\alpha = \phi^{-1}V\beta\gamma.$$
Hence
$$x = xS\alpha\phi\alpha = S.\alpha\beta\gamma,$$
and similarly for the other combinations. Thus, as we have

$$\rho S.\alpha\beta\gamma = \alpha S.\beta\gamma\rho + \beta S.\gamma\alpha\rho + \gamma S.\alpha\beta\rho,$$

we find at once

$$\phi\rho S^2.\alpha\beta\gamma = V\beta\gamma S.\beta\gamma\rho + V\gamma\alpha S.\gamma\alpha\rho + V\alpha\beta S.\alpha\beta\rho\,;$$

and the required equation may be put in the form

$$S^2.\alpha\beta\gamma = S^2.\alpha\beta\rho + S^2.\beta\gamma\rho + S^2.\gamma\alpha\rho.$$

The immediate interpretation is that *if four tetrahedra be formed by grouping, three and three, a set of semi-conjugate vector axes of an ellipsoid and any other vector of the surface, the sum of the squares of the volumes of three of these tetrahedra is equal to the square of the volume of the fourth.*

270. When the equation of a surface of the second order can be put in the form

$$S\rho\phi^{-1}\rho = 1, \quad \text{..........................} \quad (1)$$

where

$$(\phi-g)(\phi-g_1)(\phi-g_2) = 0,$$

we know that g, g_1, g_2 are the squares of the principal semi-diameters. Hence, if we put $\phi+h$ for ϕ we have a second surface, the differences of the squares of whose principal semi-axes are the same as for the first. That is,

$$S\rho(\phi+h)^{-1}\rho = 1 \quad \text{..........................} \quad (2)$$

is a surface *confocal* with (1). From this simple modification of the equation all the properties of a series of confocal surfaces may easily be deduced. We give one as an example.

271. *Any two confocal surfaces of the second order, which meet, intersect at right angles.*

For the normal to (2) is, evidently,

$$(\phi+h)^{-1}\rho;$$

and that to another of the series, if it passes through the common point whose vector is ρ, is there

$$(\phi+h_1)^{-1}\rho.$$

But $$S.(\phi+h)^{-1}\rho(\phi+h_1)^{-1}\rho = S.\rho\,\frac{1}{(\phi+h)(\phi+h_1)}\,\rho$$
$$=\frac{1}{h-h_1}\,S\rho\left((\phi+h_1)^{-1}-(\phi+h)^{-1}\right)\rho,$$

and this evidently vanishes if h and h_1 are different, as they must be unless the surfaces are identical.

272. *To find the conditions of similarity of two central surfaces of the second order.*

Referring them to their centres, let their equations be

$$\left.\begin{aligned} S\rho\phi\rho &= 1,\\ S\rho\phi'\rho &= 1.\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now the obvious conditions are that the axes of the one are proportional to those of the other. Hence, if

$$\left.\begin{aligned} g^3+m_2g^2+m_1g+m &= 0,\\ g'^3+m'_2g'^2+m'_1g'+m' &= 0,\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots \quad (2)$$

be the equations for determining the squares of the reciprocals of the semiaxes, we must have

$$\frac{m'_2}{m_2}=\mu, \qquad \frac{m'_1}{m_1}=\mu^2, \qquad \frac{m'}{m}=\mu^3, \quad\ldots\ldots\ldots\ldots \quad (3)$$

where μ is an undetermined scalar. Thus it appears that there are but two scalar conditions necessary. Eliminating μ we have

$$\frac{m'^2_2}{m^2_2} = \frac{m'_1}{m_1}, \qquad \frac{m'm'_2}{mm_2} = \frac{m'^2_1}{m^2_1}, \quad \ldots\ldots\ldots\ldots (4)$$

which are equivalent to the ordinary conditions.

273. *Find the greatest and least semi-diameters of a central plane section of an ellipsoid.*

Here

$$\left.\begin{aligned} S\rho\phi\rho &= 1 \\ S\alpha\rho &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

together represent the elliptic section; and our additional condition is that $T\rho$ is a maximum or minimum.

Differentiating the equations of the ellipse, we have

$$S\phi\rho d\rho = 0,$$

$$S\alpha d\rho = 0,$$

and the maximum condition gives

$$dT\rho = 0,$$

$$\text{or} \qquad S\rho d\rho = 0.$$

Eliminating the indeterminate vector $d\rho$ we have

$$S.\alpha\rho\phi\rho = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This shows that *the maximum or minimum vector, the normal at its extremity, and the perpendicular to the plane of section, lie in one plane.* It also shows that there are but two vectors which satisfy the conditions, and that they are perpendicular to each other, for (2) is satisfied if $\alpha\rho$ be substituted for ρ.

We have now to solve the three equations (1) and (2), to find the vectors of the two (four) points in which the ellipse (1) intersects the cone (2). We obtain at once

$$\phi\rho = xV.\phi^{-1}\alpha V\alpha\rho.$$

Operating by $S.\rho$ we have

$$1 = x\rho^2 S\alpha\phi^{-1}\alpha.$$

Hence $$\rho^2\phi\rho = \rho - \alpha\frac{S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha}$$

or $$\rho = \frac{S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha}(1-\rho^2\phi)^{-1}\alpha; \quad\ldots\ldots\ldots\ldots\ldots\ (3)$$

from which $$S.\alpha(1-\rho^2\phi)^{-1}\alpha = 0; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4)$$

a quadratic equation in ρ^2, from which the lengths of the maximum and minimum vectors are to be determined. By § 147 it may be written

$$m\rho^4 S\alpha\phi^{-1}\alpha - \rho^2 S.\alpha(m_2-\phi)\alpha + \alpha^2 = 0. \quad\ldots\ldots\ldots\ldots\ (5)$$

[If we had operated by $S.\phi^{-1}\alpha$ instead of by $S.\alpha$, we should have obtained an equation apparently different from this, but easily reducible to it. To prove their identity is a good exercise for the student.]

Substituting the values of ρ^2 given by (5) in (3) we obtain the versors of the required diameters. [The student may easily prove directly that

$$(1-\rho_1^2\phi)^{-1}\alpha \quad\text{and}\quad (1-\rho_2^2\phi)^{-1}\alpha$$

are necessarily perpendicular to each other, if both be perpendicular to α, and if ρ_1^2 and ρ_2^2 be different.]

274. By (4) of last section we see that

$$\rho_1^2\rho_2^2 = \frac{\alpha^2}{mS\alpha\phi^{-1}\alpha}.$$

Hence the area of the ellipse (1) is

$$\frac{\pi T\alpha}{\sqrt{-mS\alpha\phi^{-1}\alpha}}.$$

Also the locus of normals to all diametral sections of an ellipsoid, whose areas are equal, is the cone

$$S\alpha\phi^{-1}\alpha = C\alpha^2.$$

When the roots of (4) are equal, i. e. when

$$(m_2\alpha^2 - S\alpha\phi\alpha)^2 = 4\,m\alpha^2 S\alpha\phi^{-1}\alpha, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (5)$$

the section is a circle. It is not difficult to prove that this equation is satisfied by only two values of $U\alpha$, but another quaternion form of the equation gives the solution of this and similar problems by inspection.

275. By § 168 we may write the equation

$$S\rho\phi\rho = 1$$

in the new form

$$S.\lambda\rho\mu\rho + p\rho^2 = 1,$$

where p is a known scalar, and λ and μ are definitely known (with the exception of their tensors, whose product alone is given) in terms of the constants involved in ϕ. [The reader is referred again also to §§ 121, 122.] This may be written

$$2S\lambda\rho S\mu\rho + (p - S\lambda\mu)\rho^2 = 1. \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (1)$$

From this form it is obvious that the surface is cut by any plane perpendicular to λ or μ in a circle. For, if we put

$$S\lambda\rho = a,$$

we have

$$2aS\mu\rho + (p - S\lambda\mu)\rho^2 = 1,$$

the equation of a sphere which passes through the plane curve of intersection.

Hence λ and μ of § 168 are the values of α in equation (5) of last section.

276. *Any two circular sections of a central surface of the second order, whose planes are not parallel, lie on a sphere.*

For the equation

$$(S\lambda\rho - a)(S\mu\rho - b) = 0,$$

where a and b are any scalar constants whatever, is that of a system of two non-parallel planes, cutting the surface in circles. Eliminating the product $S\lambda\rho S\mu\rho$ between this and equation (1) of last section, there remains the equation of a sphere.

277. *To find the generating lines of a central surface of the second order.*

Let the equation be $S\rho\phi\rho = 1$;

then, if α be the vector of any point on the surface, and ϖ a vector parallel to a generating line, we must have

$$\rho = \alpha + x\varpi$$

for all values of the scalar x.

Hence $$S(\alpha + x\varpi)\phi(\alpha + x\varpi) = 1,$$

which gives the two equations

$$\left.\begin{aligned} S\alpha\phi\varpi &= 0, \\ S\varpi\phi\varpi &= 0. \end{aligned}\right\}$$

The first is the equation of a plane through the origin parallel to the tangent plane at the extremity of α, the second is the equation of the asymptotic cone. The generating lines are therefore parallel to the intersections of these two surfaces, as is well known.

From these equations we have

$$y\phi\varpi = V\alpha\varpi$$

where y is a scalar to be determined. Operating on this by $S.\beta$ and $S.\gamma$, where β and γ are any two vectors not coplanar with α, we have

$$S\varpi(y\phi\beta + V\alpha\beta) = 0, \qquad S\varpi(y\phi\gamma - V\gamma\alpha) = 0. \quad \text{.........} \ (1)$$

Hence $$S.\phi\alpha(y\phi\beta + V\alpha\beta)(y\phi\gamma - V\gamma\alpha) = 0,$$

or $$my^2 S.\alpha\beta\gamma - S\alpha\phi\alpha S.\alpha\beta\gamma = 0.$$

Thus we have the two values

$$y = \pm\sqrt{\frac{S\alpha\phi\alpha}{m}}$$

belonging to the two generating lines.

278. But by equation (1) we have

$$\begin{aligned} z\varpi &= V.(y\phi\beta + V\alpha\beta)(y\phi\gamma - V\gamma\alpha) \\ &= my^2\phi^{-1}V\beta\gamma + yV.\phi\alpha V\beta\gamma - \alpha S.\alpha V\beta\gamma; \end{aligned}$$

which, according to the sign of y, gives one or other generating line.

Here $V\beta\gamma$ may be any vector whatever, provided it is not perpendicular to a (a condition assumed in last section), and we may write for it θ.

Substituting the value of y before found, we have

$$z\varpi = Sa\phi a.\phi^{-1}\theta - aSa\theta \pm \sqrt{\frac{Sa\phi a}{m}}\, V\phi a\theta,$$

$$= V.\phi a V.a\phi^{-1}\theta \pm \sqrt{\frac{Sa\phi a}{m}}\, V\phi a\theta,$$

or, as we may evidently write it,

$$= \phi^{-1}(V.aV\phi a\theta) \pm \sqrt{\frac{Sa\phi a}{m}}\, V\phi a\theta. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Put $$\tau = V\phi a\theta,$$

and we have $$z\varpi = \phi^{-1}Va\tau \pm \sqrt{\frac{Sa\phi a}{m}}\,\tau,$$

with the condition $$S\tau\phi a = 0.$$

[Any one of these sets of values forms the complete solution of the problem; but more than one have been given, on account of their singular nature and the many properties of surfaces of the second order which immediately follow from them. It will be excellent practice for the student to show that

$$\psi\theta = U\left(V.\phi a Va\phi^{-1}\theta \pm \sqrt{\frac{Sa\phi a}{m}}\, V\phi a\theta\right)$$

is an invariant. This may most easily be done by proving that

$$V.\psi\theta\psi\theta_1 = 0 \text{ identically.}]$$

Perhaps, however, it is simpler to write a for $V.\beta\gamma$, and we thus obtain

$$z\varpi = \phi^{-1}V.aVa\phi a \pm \sqrt{\frac{Sa\phi a}{m}}\, Va\phi a.$$

[The reader need hardly be reminded that we are dealing with the *general* equation of the central surfaces of the second order—the centre being origin.]

EXAMPLES TO CHAPTER VIII.

1. Find the locus of points on the surface

$$S\rho\phi\rho = 1$$

where the generating lines are at right angles to one another.

2. Find the equation of the surface described by a straight line which revolves about an axis, which it does not meet but, with which it is rigidly connected.

3. Find the conditions that

$$S\rho\phi\rho = 1$$

may be a surface of revolution.

4. Find the equations of the right cylinders which circumscribe a given ellipsoid.

5. Find the equation of the locus of the extremities of perpendiculars to plane sections of an ellipsoid, erected at the centre, their lengths being the principal semi-axes of the sections. [Fresnel's *Wave-Surface.*]

6. The cone touching central plane sections of an ellipsoid, which are of equal area, is asymptotic to a confocal hyperboloid.

7. Find the envelop of all non-central plane sections of an ellipsoid whose area is constant.

8. Find the locus of the intersection of three planes, perpendicular to each other, and touching, respectively, each of three confocal surfaces of the second order.

9. Find the locus of the foot of the perpendicular from the centre of an ellipsoid upon the plane passing through the extremities of a set of conjugate diameters.

10. Find the points in an ellipsoid where the inclination of the normal to the radius-vector is greatest.

11. If four similar and similarly situated surfaces of the second order intersect, the planes of intersection of each pair pass through a common point.

12. If a parallelepiped be inscribed in a central surface of the second degree its edges are parallel to a system of conjugate diameters.

13. Show that there is an infinite number of sets of axes for which the Cartesian equation of an ellipsoid becomes

$$x^2+y^2+z^2 = e^2.$$

14. Find the equation of the surface of the second order which circumscribes a given tetrahedron so that the tangent plane at each angular point is parallel to the opposite face; and show that its centre is the mean point of the tetrahedron.

15. Two similar and similarly situated surfaces of the second order intersect in a plane curve, whose plane is conjugate to the vector joining their centres.

16. Find the locus of all points on

$$S\rho\phi\rho = 1,$$

where the normals meet the normal at a given point.

Also the locus of points on the surface, the normals at which meet a given line in space.

17. Normals drawn at points situated on a generating line are parallel to a fixed plane.

18. Find the envelop of the planes of contact of tangent planes drawn to an ellipsoid from points of a concentric sphere. Find the locus of the point from which the tangent planes are drawn if the envelop of the planes of contact is a sphere.

19. The sum of the reciprocals of the squares of the perpendiculars from the centre upon three conjugate tangent planes is constant.

20. Cones are drawn, touching an ellipsoid, from any two points of a similar, similarly situated, and concentric ellipsoid. Show that they intersect in two plane curves.

Find the locus of the vertices of the cones when these plane sections are at right angles to one another.

21. Find the locus of the points of contact of tangent planes which are equidistant from the centre of a surface of the second order.

22. From a fixed point A, on the surface of a given sphere, draw any chord AD; let D' be the second point of intersection of the sphere with the secant BD drawn from any point B; and take a radius vector AE, equal in length to BD', and in direction either coincident with, or opposite to, the chord AD: the locus of E is an ellipsoid, whose centre is A, and which passes through B. (Hamilton, *Elements*, p. 227.)

23. Show that the equation

$$l^2(e^2-1)(e+Saa') = (Sa\rho)^2 - 2eSa\rho Sa'\rho + (Sa'\rho)^2 + (1-e^2)\rho^2,$$

where e is a variable (scalar) parameter, represents a system of confocal surfaces. (*Ibid.* p. 644.)

24. Show that the locus of the diameters of

$$S\rho\phi\rho = 1$$

which are parallel to the chords bisected by the tangent planes to the cone

$$S\rho\psi\rho = 0$$

is the cone

$$S.\rho\phi\psi^{-1}\phi\rho = 0.$$

25. Find the equation of a cone, whose vertex is one summit of a given tetrahedron, and which passes through the circle circumscribing the opposite side.

26. Show that the locus of points on the surface

$$S\rho\phi\rho = 1,$$

the normals at which meet that drawn at the point $\rho = \varpi$, is on the cone

$$S.(\rho-\varpi)\phi\varpi\phi\rho = 0.$$

27. Find the equation of the locus of a point the square of whose distance from a given line is proportional to its distance from a given plane.

28. Show that the locus of the pole of the plane

$$S\alpha\rho = 1,$$

with respect to the surface

$$S\rho\phi\rho = 1,$$

is a sphere, if α be subject to the condition

$$S\alpha\phi^{-2}\alpha = 0.$$

29. Show that the equation of the surface generated by lines drawn through the origin parallel to the normals to

$$S\rho\phi^{-1}\rho = 1$$

along its lines of intersection with

$$S\rho(\phi+k)^{-1}\rho = 1,$$

is

$$\varpi^2 - kS\varpi(\phi+k)^{-1}\varpi = 0.$$

30. Common tangent planes are drawn to

$$2S\lambda\rho\,S\mu\rho+(p-S\lambda\mu)\rho^2 = 1, \quad \text{and} \quad T\rho = h,$$

find the value of h that the lines of contact with the former surface may be plane curves. What are they, in this case, on the sphere?

31. If tangent cones be drawn to

$$S\rho\phi^2\rho = 1,$$

from every point of

$$S\rho\phi\rho = 1,$$

the envelop of their planes of contact is

$$S\rho\phi^3\rho = 1.$$

32. Tangent cones are drawn from every point of

$$S(\rho-\alpha)\phi(\rho-\alpha) = n^2,$$

to the similar and similarly situated surface

$$S\rho\phi\rho = 1,$$

show that their planes of contact envelop the surface

$$(S\alpha\phi\rho-1)^2 = n^2\, S\rho\phi\rho.$$

33. Find the envelop of planes which touch the parabolas

$$\rho = \alpha t^2+\beta t, \quad \rho = \alpha\tau^2+\gamma\tau,$$

where α, β, γ form a rectangular system, and t and τ are scalars.

34. Find the equation of the surface on which lie the lines of contact of tangent cones drawn from a fixed point to a series of similar, similarly situated, and concentric ellipsoids.

35. Discuss the surfaces whose equations are

$$S\alpha\rho\, S\beta\rho = S\gamma\rho,$$

and $$S^2\alpha\rho+S.\alpha\beta\rho = 1.$$

36. Show that the locus of the vertices of the right cones which touch an ellipsoid is a hyperbola.

CHAPTER IX.

GEOMETRY OF CURVES AND SURFACES.

279. We have already seen (§ 31 (*l*)) that the equations

$$\rho = \phi t = \Sigma . af(t),$$

$$\text{and} \quad \rho = \phi(t, u) = \Sigma . af(t, u),$$

where a represents one of a set of given vectors, and f a scalar function of scalars t and u, represent respectively a curve and a surface. We commence the present brief Chapter with a few of the immediate deductions from these forms of expression. We shall then give a number of examples, with little attempt at systematic development or even arrangement.

280. What may be denoted by t and u in these equations is, of course, quite immaterial: but in the case of curves, considered geometrically, t is most conveniently taken as the length, s, of the curve, measured from some fixed point. In the Kinematical investigations of the next Chapter t may, with great convenience, be employed to denote *time*.

281. Thus we may write the equation of any curve in space as

$$\rho = \phi s,$$

where ϕ is a vector function of the length, s, of the curve. Of course it is only a *linear* function when the equation (as in § 31 (*h*)) represents a straight line.

282. We have also seen (§§ 38, 39) that

$$\frac{d\rho}{ds} = \frac{d}{ds}\phi s = \phi' s$$

is a vector of *unit* length in the direction of the tangent at the extremity of ρ.

At the proximate point, denoted by $s+\delta s$, this unit tangent vector becomes
$$\phi' s + \phi'' s\, \delta s + \&c.$$
But, because
$$T\phi' s = 1,$$
we have
$$S.\phi' s\, \phi'' s = 0.$$
Hence $\phi'' s$ is a vector in the osculating plane of the curve, and perpendicular to the tangent.

Also, if $\delta\theta$ be the angle between the successive tangents $\phi' s$ and $\phi' s + \phi'' s\, \delta s + \ldots\ldots$, we have
$$\mathcal{L} \frac{d\theta}{ds} = T\phi'' s;$$
so that *the tensor of $\phi'' s$ is the reciprocal of the radius of absolute curvature at the point s.*

283. Thus, if $\overline{OP} = \phi s$ be the vector of any point P of the curve, and if C be the centre of curvature at P, we have
$$\overline{PC} = -\frac{1}{\phi'' s};$$
and thus
$$\overline{OC} = \phi s - \frac{1}{\phi'' s}$$
is the equation of the locus of the centre of curvature.

Hence also
$$V.\phi' s\, \phi'' s \quad \text{or} \quad \phi' s\, \phi'' s$$
is the vector perpendicular to the osculating plane; and
$$T \frac{d}{ds} (\phi' s\, U\phi'' s)$$
is the *tortuosity* of the given curve, or the rate of rotation of its osculating plane per unit of length.

284. As an example of the use of these expressions let us *find the curve whose curvature and tortuosity are both constant.*

We have
$$\text{curvature} = T\phi'' s = T\rho'' = c.$$

Hence $$\phi' s\, \phi'' s = \rho' \rho'' = c\alpha,$$
where α is a unit vector perpendicular to the osculating plane. This gives
$$\rho' \rho''' + \rho''^2 = c\, \mathcal{L}\frac{\delta \alpha}{\delta s} = cc_1 U\rho'' = c_1 \rho'',$$
if c_1 represent the tortuosity.

Integrating we get
$$\rho' \rho'' = c_1 \rho' + \beta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
where β is a constant vector. Squaring both sides of this equation, we get
$$\begin{aligned} c^2 &= c_1^2 - \beta^2 - 2c_1 S\beta\rho' \\ &= -c_1^2 - \beta^2, \end{aligned}$$
$$\text{or} \qquad T\beta = \sqrt{c^2 + c_1^2}.$$
Multiply (1) by ρ', remembering that
$$T\rho' = 1,$$
and we obtain $$-\rho'' = -c_1 + \rho'\beta,$$
$$\text{or} \qquad \rho' = c_1 s - \rho\beta + a, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
where a is a constant quaternion. Eliminating ρ', we have
$$-\rho'' = -c_1 + c_1 s\beta - \rho\beta^2 + a\beta,$$
of which the vector part is
$$\rho'' - \rho\beta^2 = -c_1 s\beta - Va\beta.$$
The complete integral of this equation is evidently
$$\rho = \xi \cos . sT\beta + \eta \sin . sT\beta - \frac{1}{T\beta^2}(c_1 s\beta + Va\beta), \quad \ldots\ldots\ldots (3)$$
ξ and η being any two constant vectors. We have also by (2),
$$S\beta\rho = c_1 s + Sa,$$
which requires that
$$S\beta\xi = 0, \qquad S\beta\eta = 0.$$
The farther test, that
$$T\rho' = 1, \quad \text{gives us}$$
$$-1 = T\beta^2(\xi^2 \sin^2 . sT\beta + \eta^2 \cos^2 . sT\beta - 2S\xi\eta \sin . sT\beta \cos . sT\beta) - \frac{c_1^2}{c^2 + c_1^2}.$$

This requires, of course,

$$S\xi\eta = 0, \qquad T\xi = T\eta = \frac{c}{c^2+c_1^2},$$

so that (3) becomes the general equation of a helix traced on a right cylinder. (Compare § 31 (*m*).)

285. The vector perpendicular from the origin on the tangent to the curve

$$\rho = \phi s$$

is, of course,

$$\frac{1}{\rho'} V\rho'\rho, \quad \text{or} \quad \rho' V\rho\rho'$$

(since ρ' is a unit vector).

To find a common property of curves whose tangents are all equidistant from the origin.

Here

$$TV\rho\rho' = c,$$

which may be written

$$-\rho^2 - S^2\rho\rho' = c^2. \quad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

This equation shows that, as is otherwise evident, *every curve on a sphere whose centre is the origin* satisfies the condition. For obviously

$$-\rho^2 = c^2 \quad \text{gives} \quad S\rho\rho' = 0,$$

and these satisfy (1).

If $S\rho\rho'$ does not vanish, the integral of (1) is

$$\sqrt{T\rho^2 - c^2} = s, \quad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

an arbitrary constant not being necessary, as we may measure s from any point of the curve. The equation of an involute which commences at this assumed point is

$$\varpi = \rho - s\rho'.$$

This gives

$$\begin{aligned} T\varpi^2 &= T\rho^2 + s^2 - 2sS\rho\rho' \\ &= T\rho^2 + s^2 - 2s\sqrt{T\rho^2 - c^2}, \quad \text{by (1),} \\ &= c^2, \quad \text{by (2).} \end{aligned}$$

This includes *all curves whose involutes lie on a sphere about the origin.*

286. *Find the locus of the foot of the perpendicular drawn to a tangent to a right helix from a point in the axis.*

The equation of the helix is

$$\rho = \alpha \cos \frac{s}{a} + \beta \sin \frac{s}{a} + \gamma s,$$

where the vectors α, β, γ are at right angles to each other, and

$$T\alpha = T\beta = b, \quad \text{while} \quad aT\gamma = \sqrt{a^2 - b^2}.$$

The equation of the required locus is, by last section,

$$\varpi = \rho' V \rho \rho'$$

$$= \alpha \left(\cos \frac{s}{a} + \frac{a^2 - b^2}{a^3} s \sin \frac{s}{a} \right) + \beta \left(\sin \frac{s}{a} - \frac{a^2 - b^2}{a^3} s \cos \frac{s}{a} \right) + \gamma \frac{b^2}{a^2} s.$$

This curve lies on the hyperboloid whose equation is

$$S^2 \alpha \varpi + S^2 \beta \varpi - a^2 S^2 \gamma \varpi = b^4,$$

as the reader may easily prove for himself.

287. *To find the least distance between consecutive tangents to a tortuous curve.*

Let one tangent be

$$\varpi = \rho + x\rho',$$

then a consecutive one, at a distance δs along the curve, is

$$\varpi = \rho + \rho' \delta s + \rho'' \frac{\delta s^2}{1.2} + \&c. + y \left(\rho' + \rho'' \delta s + \rho''' \frac{\delta s^2}{1.2} + \ldots \right).$$

The magnitude of the least distance between these lines is, by §§ 203, 210,

$$S.\left(\rho' \delta s + \rho'' \frac{\delta s^2}{1.2} + \rho''' \frac{\delta s^3}{1.2.3} + \ldots \right) UV.\rho' \left(\rho' + \rho'' \delta s + \rho''' \frac{\delta s^2}{1.2} + \ldots \right)$$

$$= \frac{-\dfrac{\delta s^4}{12} S.\rho' \rho'' \rho'''}{TV\rho' \rho'' \delta s},$$

if we neglect terms of higher orders.

It may be written, since $\rho'\rho''$ is a vector, and $T\rho' = 1$,

$$\frac{\delta s^3}{12} S.U\rho'' V\rho'\rho'''.$$

But $$\frac{\delta UV\rho'\rho''}{UV\rho'\rho''} = V\frac{V\rho'\rho'''}{V\rho'\rho''}\delta s = \frac{\delta s}{\rho''^2}\rho' S.\rho'\rho''\rho'''.$$

Hence $$\frac{\delta s}{T\rho''} S.U\rho'' V\rho'\rho'''$$

is the small angle, $\delta\phi$, between the two successive positions of the osculating plane. [See also § 283.]

Thus the shortest distance between two consecutive tangents is expressed by the formula

$$\frac{\delta\phi\,\delta s^2}{12r},$$

where $r, = \frac{1}{T\rho''}$, is the radius of absolute curvature of the tortuous curve.

288. Let us recur for a moment to the equation of the parabola (§ 31 (*f*))

$$\rho = \alpha t + \frac{\beta t^2}{2}.$$

Here $$\rho' = (\alpha + \beta t)\frac{dt}{ds},$$

whence, if we assume $S\alpha\beta = 0$,

$$\frac{ds}{dt} = \sqrt{-\alpha^2 - \beta^2 t^2},$$

from which the length of the arc of the curve can be derived in terms of t by integration.

Again, $$\rho'' = (\alpha + \beta t)\frac{d^2t}{ds^2} + \beta\left(\frac{dt}{ds}\right)^2.$$

But $$\frac{d^2t}{ds^2} = \frac{d}{ds}\cdot\frac{1}{T(\alpha+\beta t)} = +\frac{dt}{ds}\,\frac{S.\beta(\alpha+\beta t)}{T(\alpha+\beta t)^3}.$$

Hence $$\rho'' = \frac{(\alpha+\beta t)\,V\alpha\beta}{T(\alpha+\beta t)^4},$$

and therefore, for the vector of the centre of curvature we have (§ 283),

$$\varpi = \alpha t + \frac{\beta t^2}{2} - (\alpha^2 + \beta^2 t^2)^2 (-\beta\alpha^2 + \alpha\beta^2 t)^{-1},$$

$$= \beta\left(\frac{3t^2}{2} + \frac{\alpha^2}{\beta^2}\right) - \alpha\frac{t^3\beta^2}{\alpha^2};$$

which is the quaternion equation of the evolute.

289. One of the simplest forms of the equation of a tortuous curve is

$$\rho = \alpha t + \frac{\beta t^2}{2} + \frac{\gamma t^3}{6},$$

where α, β, γ are any three non-coplanar vectors, and the numerical factors are introduced for convenience. This curve lies on a parabolic cylinder whose generating lines are parallel to γ; and also on cylinders whose bases are a cubical and a semi-cubical parabola, their generating lines being parallel to β and α respectively. We have by the equation of the curve

$$\rho' = \left(\alpha + \beta t + \frac{\gamma t^2}{2}\right)\frac{dt}{ds},$$

from which, by $T\rho' = 1$, the length of the curve can be found in terms of t; and

$$\rho'' = \left(\alpha + \beta t + \frac{\gamma t^2}{2}\right)\frac{d^2 t}{ds^2} + (\beta + \gamma t)\left(\frac{dt}{ds}\right)^2,$$

from which ρ'' can be expressed in terms of s. The investigation of various properties of this curve is very easy, and will be of great use to the student.

It is to be observed that in this equation t cannot stand for s, the length of the curve. Such an equation as

$$\rho = \alpha s + \beta s^2 + \gamma s^3,$$

or even the simpler form

$$\rho = \alpha s + \beta s^2,$$

involves an absurdity.

290. The equation $\rho = \phi^t \epsilon,$

where ϕ is a self-conjugate linear and vector function, t a scalar variable, and ϵ an arbitrary vector constant, denotes a curious class of curves.

We have at once

$$\frac{d\rho}{dt} = \phi^t \log \phi \epsilon,$$

where $\log \phi$ is another self-conjugate linear and vector function, which we may denote by χ. These functions are obviously commutative, as they have the same principal set of rectangular vectors, hence we may write

$$\frac{d\rho}{dt} = \chi\rho,$$

which of course gives

$$\frac{d^2\rho}{dt^2} = \chi^2\rho, \text{ \&c.}$$

As a verification, we should have

$$\begin{aligned} \phi^{t+\delta t}\epsilon &\doteq \rho + \frac{d\rho}{dt}\delta t + \frac{d^2\rho}{dt^2}\frac{\delta t^2}{1.2} + \text{\&c.} \\ &= \left(1 + \delta t\chi + \frac{\delta t^2}{1.2}\chi^2 + \ldots\ldots\right)\rho \\ &= e^{\delta t \chi}\rho, \end{aligned}$$

where e is the base of Napier's Logarithms.

This is obviously true if

$$\phi^{\delta t} = e^{\delta t \chi},$$

$$\text{or} \qquad \phi = e^{\chi},$$

$$\text{or} \quad \log \phi = \chi,$$

which is our assumption.

[The above process is, at first sight, rather startling, but the student may easily verify it by writing, in accordance with the results of Chapter V,

$$\phi\epsilon = -g_1 a Sa\epsilon - g_2 \beta S\beta\epsilon - g_3 \gamma S\gamma\epsilon,$$

whence $$\phi^t\epsilon = -g_1^t a Sa\epsilon - g_2^t \beta S\beta\epsilon - g_3^t \gamma S\gamma\epsilon.$$

He will find at once

$$\chi\epsilon = -\log g_1\, \alpha S\alpha\epsilon - \log g_2\, \beta S\beta\epsilon - \log g_3\, \gamma S\gamma\epsilon,$$

and the results just given follow immediately.]

291. That the equation

$$\rho = \phi(t, u) = \Sigma . \alpha f(t, u)$$

represents a surface is obvious from the fact that it becomes the equation of a definite curve whenever *either* t or u has a particular value assigned to it. Hence the equation at once furnishes us with two systems of curves, lying wholly on the surface, and such that one of each system can, in general, be drawn through any assigned point on the surface. Tangents drawn to these curves at their point of intersection must, of course, lie in the tangent plane, whose equation we have thus the means of forming.

292. By the equation we have

$$d\rho = \left(\frac{d\phi}{dt}\right) dt + \left(\frac{d\phi}{du}\right) du,$$

where the brackets are inserted to show that these quantities are partial differential coefficients. If we write this as

$$d\rho = \phi_t' dt + \phi_u' du,$$

the normal to the tangent plane is evidently

$$V\phi_t'\phi'_u,$$

and the equation of that plane

$$S.(\varpi - \phi)\phi_t' \phi_u' = 0.$$

293. As a simple example, suppose a straight line to move along a fixed straight line, remaining always perpendicular to it, while rotating about it through an angle proportional to the space it has advanced; the equation of the ruled surface described will evidently be

$$\rho = \alpha t + u(\beta \cos t + \gamma \sin t), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where α, β, γ are rectangular vectors, and

$$T\beta = T\gamma.$$

This surface evidently intersects the right cylinder

$$\rho = a(\beta \cos t + \gamma \sin t) + v\alpha,$$

in a helix (§§ 31 (*m*), 284) whose equation is

$$\rho = \alpha t + a(\beta \cos t + \gamma \sin t).$$

These equations illustrate very well the remarks made in §§ 31 (*l*), 291, as to the curves or surfaces represented by a vector equation according as it contains one or two scalar variables.

From (1) we have

$$d\rho = [\alpha - u(\beta \sin t - \gamma \cos t)]\, dt + (\beta \cos t + \gamma \sin t)\, du,$$

so that the normal at the extremity of ρ is

$$T\alpha(\gamma \cos t - \beta \sin t) - u\, T\beta^2\, U\alpha.$$

Hence, as we proceed along a generating line of the surface, for which t is constant, we see that the direction of the normal changes. This, of course, proves that the surface is not developable.

294. Hence the criterion for a developable surface is that if it be expressed by an equation of the form

$$\rho = \phi t + u\psi t,$$

where ϕt and ψt are vector functions, we must have the *direction* of the normal

$$V\{\phi' t + u\psi' t\}\, \psi t$$

independent of u.

This requires either $V\psi t\, \psi' t = 0,$

which would reduce the surface to a cylinder, all the generating lines being parallel to each other; or

$$V\phi' t\, \psi t = 0.$$

This is the criterion we seek, and it shows that we may write, for a developable surface in general, the equation

$$\rho = \phi t + u\phi' t. \qquad \text{.............................. (1)}$$

Evidently $$\rho = \phi t$$
is a curve (generally tortuous) and $\phi' t$ is a tangent vector. Hence *a developable surface is the locus of all tangent lines to a tortuous curve.*

Of course the tangent plane to the surface is the osculating plane at the corresponding point of the curve; and this is indicated by the fact that the normal to (1) is parallel to

$$V\phi' t \phi'' t. \quad \text{(See § 283.)}$$

295. A *Geodetic* line is a curve drawn on a surface so that its osculating plane at any point contains the normal to the surface. Hence, if ν be the normal at the extremity of ρ, ρ' and ρ'' the first and second differentials of the vector of the geodetic,

$$S.\nu\rho'\rho'' = 0,$$

which may be easily transformed into

$$V.\nu dU\rho' = 0.$$

296. In the sphere $T\rho = a$ we have

$$\nu \parallel \rho,$$

hence $$S.\rho\rho'\rho'' = 0,$$

which shows of course that ρ is confined to a plane passing through the origin, the centre of the sphere.

For a formal proof, we may proceed as follows—

The above equation is equivalent to the three

$$S\theta\rho = 0, \qquad S\theta\rho' = 0, \qquad S\theta\rho'' = 0,$$

from which we see at once that θ is a constant vector, and therefore the first expression, which includes the others, is the complete integral.

Or we may proceed thus—

$$0 = -\rho S.\rho\rho'\rho'' + \rho'' S.\rho^2\rho' = V.V\rho\rho' V\rho\rho'' = V.V\rho\rho' dV\rho\rho',$$

whence by § 133 (2) we have at once

$$UV\rho\rho' = \text{const.} = \theta \text{ suppose,}$$

which gives the same results as before.

297. In any cone we have, of course,

$$S\nu\rho = 0,$$

since ρ lies in the tangent plane. But we have also

$$S\nu\rho' = 0.$$

Hence, by the general equation of § 295, eliminating ν we get

$$0 = ds S.\rho\rho' V\rho'\rho'' = S\rho dU\rho' \text{ by § 133 (2).}$$

Integrating

$$C = S\rho U\rho' - \int S d\rho U\rho' = S\rho U\rho' + \int T d\rho.$$

The interpretation of this is, that the length of any arc of the geodetic is equal to the projection of the side of the cone (drawn to its extremity) upon the tangent to the geodetic. In other words, *when the cone is developed the geodetic becomes a straight line.* A similar result may easily be obtained for the geodetic lines on any developable surface whatever.

298. *To find the shortest line connecting two points on a given surface.*

Here $\int T d\rho$ is to be a minimum, subject to the condition that $d\rho$ lies in the given surface.

$$\text{Now} \quad \delta \int T d\rho = \int \delta T d\rho = -\int \frac{S d\rho\, d\delta\rho}{T d\rho} = -\int S.U d\rho\, d\delta\rho$$

$$= -[S.U d\rho\, \delta\rho] + \int S.\delta\rho\, d\, U d\rho,$$

where the term in brackets vanishes at the limits, as the extreme points are fixed, and therefore $\delta\rho = 0$.

Hence our only conditions are

$$\int S.\delta\rho d U d\rho = 0, \quad \text{and} \quad S\nu\delta\rho = 0, \quad \text{giving}$$

$$V.\nu d U d\rho = 0, \quad \text{as in § 295.}$$

If the extremities of the curve are not given, but are to lie on given curves, we must refer to the integrated portion of the expression for the variation of the length of the arc. And its form

$$S.Ud\rho\,\delta\rho$$

shows that the shortest line cuts each of the given curves at right angles.

299. The osculating plane of the curve

$$\rho = \phi t$$

is

$$S.\phi' t\,\phi'' t(\varpi-\rho) = 0, \qquad \ldots\ldots\ldots\ldots (1)$$

and is, of course, the tangent plane to the surface

$$\rho = \phi t + u\phi' t. \qquad \ldots\ldots\ldots\ldots (2)$$

Let us attempt the converse of the process we have, so far, pursued, and endeavour to find (2) as the envelop of the variable plane (1).

Differentiating (1) with respect to t only, we have

$$S.\phi'\phi'''(\varpi-\rho) = 0.$$

By this equation, combined with (1), we have

$$\varpi-\rho \parallel V.V\phi'\phi''V\phi'\phi''' \parallel \phi',$$

or

$$\varpi = \rho + u\phi' = \phi + u\phi',$$

which is equation (2).

300. This leads us to the consideration of envelops generally, and the process just employed may easily be extended to the problem of *finding the envelop of a series of surfaces whose equation contains one scalar parameter.*

When the given equation is a scalar one, the process of finding the envelop is precisely the same as that employed in ordinary Cartesian geometry, though the work is often shorter and simpler.

If the equation be given in the form

$$\rho = \psi(t, u, v),$$

where ψ is a vector function, t and u the scalar variables for any one surface, v the scalar parameter, we have for a proximate surface

$$\rho_1 = \psi(t_1, u_1, v_1) = \rho + \psi'_t \delta t + \psi'_u \delta u + \psi'_v \delta v.$$

Hence at all points on the intersection of two successive surfaces of the series we have

$$\psi'_t \delta t + \psi'_u \delta u + \psi'_v \delta v = 0,$$

which is equivalent to the following scalar equation connecting the quantities t, u, and v;

$$S.\psi'_t \psi'_u \psi'_v = 0.$$

This equation, along with

$$\rho = \psi(t, u, v),$$

enables us to eliminate t, u, v, and the resulting scalar equation is that of the required envelop.

301. As an example, let us find the envelop of the osculating plane of a tortuous curve. Here the equation of the plane is (§ 299),

$$S.(\varpi - \rho)\phi' t\, \phi'' t = 0,$$

or $$\varpi = \phi t + x\phi' t + y\phi'' t = \psi(x, y, t),$$

if $$\rho = \phi t$$

be the equation of the curve.

Our condition is, by last section,

$$S.\psi'_x \psi'_y \psi'_t = 0,$$

or $$S.\phi' t\, \phi'' t\, [\phi' t + x\phi'' t + y\phi''' t] = 0,$$

or $$yS.\phi' t\, \phi'' t\, \phi''' t = 0.$$

Now the scalar factor cannot vanish, unless the given curve be plane, so that

$$y = 0,$$

and the envelop is

$$\varpi = \phi t + x\phi' t$$

e developable surface, of which the given curve is the edge regression, as in § 299.

302. When the equation contains two scalar parameters its differential coefficients with respect to them must vanish, and we have thus three equations from which to eliminate two numerical quantities.

A very common form in which these two parameters appear in quaternions is that of an unknown unit-vector. In this case the problem may be thus stated—*Find the envelop of the surface whose scalar equation is*

$$F(\rho, \alpha) = 0,$$

where α *is subject to the one condition*

$$T\alpha = 1.$$

Differentiating with respect to α alone, we have

$$S\nu d\alpha = 0, \qquad S\alpha d\alpha = 0,$$

where ν is a known vector function of ρ and α. Since $d\alpha$ may have any of an infinite number of values, these equations show that

$$V\alpha\nu = 0.$$

This is equivalent to two scalar conditions only, and these, in addition to the two given scalar equations, enable us to eliminate α.

With the brief explanation we have given, and the examples which follow, the student will easily see how to deal with any other set of data he may meet with in a question of envelops.

303. *Find the envelop of a plane whose distance from the origin is constant.*

Here $$S\alpha\rho = -c,$$
with the condition $$T\alpha = 1.$$

Hence, by last section,

$$V\rho\alpha = 0,$$

and therefore $$\rho = c\alpha,$$

or $$T\rho = c,$$

the sphere of radius c, as was to be expected.

If we seek the *envelop of those only of the planes which are parallel to a given vector* β, we have the additional relation

$$S\alpha\beta = 0.$$

In this case the three differentiated equations are

$$S\rho d\alpha = 0, \qquad S\alpha d\alpha = 0, \qquad S\beta d\alpha = 0,$$

and they give $\quad S.\alpha\beta\rho = 0.$

Hence $\quad \alpha = U.\beta V\beta\rho,$

and the envelop is $\quad TV\beta\rho = cT\beta,$

the circular cylinder of radius c and axis coinciding with β.

By putting $S\alpha\beta = e$, where e is a constant different from zero, we pick out all the planes of the series which have a definite inclination to β, and of course get as their envelop a right cone.

304. The equation

$$S^2\alpha\rho + 2S.\alpha\beta\rho = b$$

represents a parabolic cylinder, whose generating lines are parallel to the vector $\alpha V\alpha\beta$. For it is not altered by increasing ρ by the vector $x\alpha V\alpha\beta$; also it cuts planes perpendicular to α in one line, and planes perpendicular to $V\alpha\beta$ in two parallel lines. Its form and position of course depend upon the values of α, β, and b. *It is required to find its envelop* if β and b be constant, and α be subject to the one scalar condition

$$T\alpha = 1.$$

The process of § 302 gives, by inspection,

$$\rho S\alpha\rho + V\beta\rho = x\alpha.$$

Operating by $S.\alpha$, we get

$$S^2\alpha\rho + S.\alpha\beta\rho = -x,$$

which gives $\quad S.\alpha\beta\rho = x + b.$

But, by operating successively by $S.V\beta\rho$ and by $S.\rho$, we have

$$(V\beta\rho)^2 = xS.\alpha\beta\rho,$$

and $\quad (\rho^2 - x)S\alpha\rho = 0.$

Omitting, for the present, the factor $Sa\rho$, these three equations give, by elimination of x and a,

$$(V\beta\rho)^2 = \rho^2(\rho^2+b),$$

which is the equation of the envelop required.

This is evidently a surface of revolution of the fourth order whose axis is β; but, to get a clearer idea of its nature, put

$$c^2\rho^{-1} = \varpi,$$

and the equation becomes

$$(V\beta\varpi)^2 = c^4+b\varpi^2,$$

which is obviously a surface of revolution of the second degree, referred to its centre. Hence the required envelop is the *reciprocal* of such a surface, in the sense that *the rectangle under the lengths of condirectional radii of the two is constant.*

We have a curious particular case if the constants are so related that

$$b+\beta^2 = 0,$$

for then the envelop breaks up into the two equal spheres, touching each other at the origin,

$$\rho^2 = \pm S\beta\rho,$$

while the corresponding surface of the second order becomes the two parallel planes

$$S\beta\varpi = \pm c^2.$$

305. The particular solution above met with, viz.

$$Sa\rho = 0,$$

limits the original problem, which now becomes one of finding the envelop of a line instead of a surface. In fact this equation, taken in conjunction with that of the parabolic cylinder, belongs to that generating line of the cylinder which is the locus of the vertices of the principal parabolic sections.

Our equations become

$$\begin{aligned} 2Sa\beta\rho &= b, \\ Sa\rho &= 0, \end{aligned}$$

$$Ta = 1;$$

whence $V\beta\rho = xa,$ giving

$$x = -S.a\beta\rho = -\frac{b}{2},$$

and thence $$TV\beta\rho = \frac{b}{2};$$

so that the envelop is a circular cylinder whose axis is β. [It is to be remarked that the equations above require that

$$Sa\beta = 0,$$

so that the problem now solved is merely that of *the envelop of a parabolic cylinder which rotates about its focal line.* This discussion has only been entered into for the sake of explaining a peculiarity in a former result, because of course the present results can be obtained immediately by an exceedingly simple process.]

306. The equation
$$Sa\rho\, S.a\beta\rho = a^2,$$
with the condition $$Ta = 1,$$

represents a series of hyperbolic cylinders. *It is required to find their envelop.*

As before, we have

$$\rho S.a\beta\rho + V\beta\rho\, Sa\rho = xa,$$

which by operating by $S.a$, $S.\rho$, and $S.V\beta\rho$, gives

$$2a^2 = -x,$$

$$\rho^2 S.a\beta\rho = x\, Sa\rho,$$

$$(V\beta\rho)^2\, Sa\rho = x\, S.a\beta\rho.$$

Eliminating a and x we have, as the equation of the envelop,

$$\rho^2 (V\beta\rho)^2 = 4a^4.$$

Comparing this with the equations

$$\rho^2 = -2a^2,$$

and $$(V\beta\rho)^2 = -2a^2,$$

which represent a sphere and one of its circumscribing cylinders, we see that, if condirectional radii of the three surfaces be drawn from the origin, that of the new surface is a geometric mean between those of the two others.

307. *Find the envelop of all spheres which touch one given line and have their centres in another.*

Let $$\rho = \beta + y\gamma$$
be the line touched by all the spheres, and let $x\alpha$ be the vector of the centre of any one of them, the equation is (by § 200, or § 201)
$$\gamma^2(\rho - x\alpha)^2 = -(V.\gamma(\beta - x\alpha))^2,$$
or, putting for simplicity, but without loss of generality,
$$T\gamma = 1, \quad S\alpha\beta = 0, \quad S\beta\gamma = 0,$$
so that β is the least vector distance between the given lines,
$$(\rho - x\alpha)^2 = (\beta - x\alpha)^2 + x^2 S^2\alpha\gamma,$$
and, finally, $$\rho^2 - \beta^2 - 2x\, S\alpha\rho = x^2\, S^2\alpha\gamma.$$
Hence, by § 300, $$-2\, S\alpha\rho = .2x S^2\alpha\gamma.$$
[This gives no definite envelop if
$$S\alpha\gamma = 0,$$
i. e. if the line of centres is perpendicular to the line touched by all the spheres.]

Eliminating x, we have for the equation of the envelop
$$S^2\alpha\rho + S^2\alpha\gamma\,(\rho^2 - \beta^2) = 0,$$
which denotes a surface of revolution of the second degree, whose axis is α.

Since, from the form of the equation, $T\rho$ may have any magnitude not less than $T\beta$, and since the section by the plane
$$S\alpha\rho = 0$$
is a real circle, on the sphere
$$\rho^2 - \beta^2 = 0,$$
the surface is a hyperboloid of one sheet.

[It will be instructive to the student to find the signs of the values of g_1, g_2, g_3 as in § 165, and thence to prove the above conclusion.]

308. As a final example let us find the envelop of the hyperbolic cylinder

$$S\alpha\rho \, S\beta\rho - c = 0,$$

where the vectors α and β are subject to the conditions

$$T\alpha = T\beta = 1,$$

$$S\alpha\gamma = 0, \qquad S\beta\delta = 0.$$

[It will be easily seen that two of the six scalars involved in α, β still remain as variable parameters.]

We have $$S\alpha d\alpha = 0, \qquad S\gamma d\alpha = 0,$$

so that $$d\alpha = x V\alpha\gamma.$$

Similarly $$d\beta = y V\beta\delta.$$

But, by the equation of the cylinders,

$$S\alpha\rho \, S\rho d\beta + S\rho d\alpha \, S\beta\rho = 0,$$

or $$y \, S\alpha\rho \, S.\beta\delta\rho + x \, S.\alpha\gamma\rho \, S\beta\rho = 0.$$

Now by the nature of the given equation, neither $S\alpha\rho$ nor $S\beta\rho$ can vanish, so that the independence of $d\alpha$ and $d\beta$ requires

$$S.\alpha\gamma\rho = 0, \qquad S.\beta\delta\rho = 0.$$

Hence $$\alpha = U.\gamma V\gamma\rho, \quad \beta = U.\delta V\delta\rho,$$

and the envelop is

$$T.V\gamma\rho \, V\delta\rho - cT\gamma\delta = 0,$$

a surface of the fourth order, which may be constructed by laying off mean proportionals between the lengths of condirectional radii of two equal right cylinders whose axes meet in the origin.

309. We may now easily see the truth of the following general statement.

Suppose the given equation of the series of surfaces, whose envelop is required, to contain m vector, and n scalar, parameters; and that the latter are subject to p vector, and q scalar, conditions.

In all there are $3m+n$ scalar parameters, subject to $3p+q$ scalar conditions.

That there may be an envelop we must therefore in general have

$$(3m+n)-(3p+q) = 1, \quad \text{or} \quad = 2.$$

In the former case the enveloping surface is given as the locus of a series of *curves*, in the latter a series of *points*.

Differentiation of the equations gives us $3p+q+1$ equations, linear and homogeneous in the $3m+n$ differentials of the scalar parameters, so that by the elimination of these we have *one* final scalar equation in the first case, *two* in the second; and thus in each case we have just equations enough to eliminate all the arbitrary parameters.

310. *To find the locus of the foot of the perpendicular drawn from the origin to a tangent plane to any surface.*

If

$$S\nu d\rho = 0$$

be the differentiated equation of the surface, the equation of the tangent plane is

$$S(\varpi-\rho)\nu = 0.$$

We may introduce the condition

$$S\nu\rho = 1,$$

which in general alters the tensor of ν, so that ν^{-1} becomes the required vector perpendicular, as it satisfies the equation

$$S\varpi\nu = 1.$$

It remains that we eliminate ρ between the equation of the given surface, and the vector equation

$$\varpi = \nu^{-1}.$$

The result is the scalar equation (in ϖ) required.

For example, if the given surface be the ellipsoid

$$S\rho\phi\rho = 1,$$

we have $$\varpi^{-1} = \nu = \phi\rho,$$

so that the required equation is

$$S\varpi^{-1}\phi^{-1}\varpi^{-1} = 1,$$

or $$S\varpi\phi^{-1}\varpi = \varpi^4,$$

which is Fresnel's *Surface of Elasticity*. (§ 263.)

311. *To find the reciprocal of a given surface with respect to the unit sphere whose centre is the origin.*

With the condition

$$S\rho\nu = 1,$$

of last section, we see that $-\nu$ is the vector of the pole of the tangent plane

$$S(\varpi-\rho)\nu = 0.$$

Hence we must put $\varpi = -\nu$,

and eliminate ρ by the help of the equation of the given surface.

Take the ellipsoid of last section, and we have

$$\varpi = -\phi\rho,$$

so that the reciprocal surface is represented by

$$S\varpi\phi^{-1}\varpi = 1.$$

It is obvious that the former ellipsoid can be reproduced from this by a second application of the process.

And the property is general, for

$$S\rho\nu = 1$$

gives, by differentiation, and attention to the condition

$$S\nu d\rho = 0,$$

the new relation $$S\rho d\nu = 0,$$

so that ρ and ν are corresponding vectors of the two surfaces: either being that of the pole of a tangent plane drawn at the extremity of the other.

312. If the given surface be a cone with its vertex at the origin, we have a peculiar case. For here every tangent plane passes through the origin, and therefore the required locus is wholly at an infinite distance. The difficulty consists in $S\rho\nu$ becoming in this case a numerical multiple of the quantity which is equated to zero in the equation of the cone, so that of course we cannot put as above

$$S\rho\nu = 1.$$

313. The properties of the normal vector ν enable us to write the partial differential equations of families of surfaces in a very simple form.

Thus the distinguishing property of *Cylinders* is that all their generating lines are parallel. Hence all positions of ν must be parallel to a given plane—or

$$S\alpha\nu = 0,$$

which is the quaternion form of the well-known equation

$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0.$$

To integrate it, remember that we have always

$$S\nu d\rho = 0,$$

and that as ν is perpendicular to α it may be expressed in terms of any two vectors, β and γ, each perpendicular to α.

Hence $$\nu = x\beta + y\gamma,$$

and $$xS\beta d\rho + yS\gamma d\rho = 0.$$

This shows that $S\beta\rho$ and $S\gamma\rho$ are together constant or together variable, so that

$$S\beta\rho = f(S\gamma\rho),$$

where f is any scalar function whatever.

314. In *Surfaces of Revolution* the normal intersects the axis. Hence, taking the origin in the axis α, we have

$$S.\alpha\rho\nu = 0,$$

or $$\nu = x\alpha + y\rho.$$

Hence $$xS\alpha d\rho + yS\rho d\rho = 0,$$

whence the integral $$T\rho = f(S\alpha\rho).$$

The more common form, which is easily derived from that just written, is $$TV\alpha\rho = F(S\alpha\rho).$$

In *Cones* we have $$S\nu\rho = 0,$$

and therefore

$$S\nu d\rho = S.\nu(T\rho dU\rho + U\rho dT\rho) = T\rho S\nu dU\rho.$$

Hence $$S\nu dU\rho = 0,$$

so that ν must be a function of $U\rho$, and therefore the integral is

$$f(U\rho) = 0,$$

which simply expresses the fact that the equation does not involve the tensor of ρ.

315. *If equal lengths be laid off on the normals drawn to any surface, the new surface formed by their extremities is normal to the same lines.*

For we have $$\varpi = \rho + aU\nu,$$

and $$S\nu d\varpi = S\nu d\rho + aS\nu dU\nu = 0,$$

which proves the proposition.

Take, for example, the surface

$$S\rho\phi\rho = 1;$$

the above equation becomes

$$\varpi = \rho + \frac{a\phi\rho}{T\phi\rho};$$

so that $$\rho = \left(\frac{a\phi}{T\phi\rho} + 1\right)^{-1}\varpi,$$

and the equation of the new surface is to be found by eliminating $\frac{a}{T\phi\rho}$ (written x) between the equations

$$1 = S.(x\phi+1)^{-1}\varpi\phi(x\phi+1)^{-1}\varpi,$$

and $$-\frac{a^2}{x^2} = S.\phi(x\phi+1)^{-1}\varpi\phi(x\phi+1)^{-1}\varpi.$$

316. It appears from last section that if one orthogonal surface can be drawn cutting a given system of straight lines, an indefinitely great number may be drawn: and that the portions of these lines intercepted between any two selected surfaces of the series are all equal.

Let $$\rho = \sigma + x\tau,$$

where σ and τ are vector functions of ρ, and x is any scalar, be the general equation of a system of lines: we have

$$S\tau d\rho = 0 = S(\rho-\sigma)d\rho$$

as the differentiated equation of the series of orthogonal surfaces, if it exist. Hence the following problem.

317. It is required *to find the criterion of integrability of the equation*

$$S\nu d\rho = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

as the complete differential of the equation of a series of surfaces.

Hamilton has given (*Elements*, p. 702) an extremely elegant solution of this problem, by means of the properties of linear and vector functions. We adopt a different and less simple process, on account of some results it offers which will be useful to us in the next Chapter; and also because it will show the student the connection of our methods with those of ordinary differential equations.

If we assume $$F\rho = C$$

to be the integral, and apply to it the very singular operator devised by Hamilton,

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

we have $$\nabla F\rho = i\frac{dF}{dx} + j\frac{dF}{dy} + k\frac{dF}{dz}.$$

But $\rho = ix + jy + kz,$

whence $d\rho = idx + jdy + kdz,$

and $0 = dF = \frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = -S\nabla F\,d\rho.$

Comparing with the given equation, we see that the latter represents a series of surfaces if ν, or a scalar multiple of it, can be expressed as ∇F.

If $\nu = \nabla F,$

we have $\nabla\nu = \nabla^2 F = -\left(\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}\right),$

a well-known and most important expression, to which we shall return in next Chapter. Meanwhile we need only remark that the last-written quantities are necessarily scalars, so that the only requisite condition of the integrability of (1) is

$$V\nabla\nu = 0. \qquad (2)$$

If ν do not satisfy this criterion, it may when multiplied by a scalar. Hence the farther condition

$$V\nabla(w\nu) = 0,$$

which may be written

$$V\nu\nabla w + wV\nabla\nu = 0. \qquad (3)$$

This requires that

$$S\nu\nabla\nu = 0. \qquad (4)$$

If then (2) be not satisfied, we must try (4). If (4) be satisfied w will be found from (3); and in either case (1) is at once integrable.

[If we put $d\nu = \phi\,d\rho$

where ϕ is a linear and vector function, not necessarily self-conjugate, we have

$$V\nabla\nu = V\left(i\frac{d\nu}{dx} + \dots\right) = V(i\phi i + \dots) = -\epsilon,$$

by § 173. Thus, if ϕ be self-conjugate, $\epsilon = 0$, and the criterion

(2) is satisfied. If ϕ be not self-conjugate we have by (4) for the criterion

$$S\epsilon\nu = 0.$$

These results accord with Hamilton's, lately referred to, but the mode of obtaining them is quite different from his.]

318. As a simple example let us first take *lines diverging from a point.* Here $\nu \| \rho$, and we see that if $\nu = \rho$

$$\nabla\nu = -3,$$

so that (2) is satisfied. And the equation is

$$S\rho d\rho = 0,$$

whose integral $$T\rho = C$$

gives a series of concentric spheres.

Lines perpendicular to, and intersecting, a fixed line.

If α be the fixed line, β any of the others, we have

$$S.\alpha\beta\rho = 0, \qquad S\alpha\beta = 0, \qquad S\beta d\rho = 0.$$

Here $$\nu \| \alpha V\alpha\rho,$$

and therefore equal to it, because (2) is satisfied.

Hence $$S.d\rho\alpha V\alpha\rho = 0,$$

or $$\alpha^2 S\rho d\rho - S\alpha\rho S\alpha d\rho = 0,$$

whose integral is the equation of a series of right cylinders

$$\alpha^2\rho^2 - S^2\alpha\rho = T^2 V\alpha\rho = C.$$

319. *To find the orthogonal trajectories of a series of circles whose centres are in, and their planes perpendicular to, a given line.*

Let α be a unit-vector in the direction of the line, then one of the circles has the equations

$$\left.\begin{aligned} T\rho &= C, \\ S\alpha\rho &= C', \end{aligned}\right\}$$

where C and C' are any constant scalars whatever.

Hence, for the required surfaces

$$\nu \parallel d_1\rho \parallel V\alpha\rho,$$

where $d_1\rho$ is an element of one of the circles, ν the normal to the orthogonal surface. Now let $d\rho$ be an element of a tangent to the orthogonal surface, and we have

$$S\nu d\rho = S.\alpha\rho d\rho = 0.$$

This shows that $d\rho$ is in the same plane as α and ρ, i. e. that the orthogonal surfaces are planes passing through the common axis.

[To integrate the equation

$$S.\alpha\rho d\rho = 0$$

evidently requires, by § 317, the introduction of a factor. For

$$\begin{aligned} V\nabla V\alpha\rho &= V(iV\alpha i + jV\alpha j + kV\alpha k) \\ &= 2\alpha, \end{aligned}$$

so that the first criterion is not satisfied. But

$$S.V\alpha\rho V\nabla V\alpha\rho = 2S.\alpha V\alpha\rho = 0,$$

so that the second criterion holds. It gives, by (3) of § 317,

$$V.\nabla w V\alpha\rho + 2w\alpha = 0,$$

or

$$\rho S\alpha\nabla w - \alpha S\rho\nabla w + 2w\alpha = 0.$$

That is

$$\left.\begin{aligned} S\alpha\nabla w &= 0, \\ S\rho\nabla w &= 2w. \end{aligned}\right\}$$

These equations are satisfied by

$$w = \frac{1}{V^2\alpha\rho}.$$

But a simpler mode of integration is easily seen. Our equation may be written

$$S.\alpha\rho d\rho = 0,$$

$$0 = S.\alpha V\frac{d\rho}{\rho} = S\alpha\frac{dU\rho}{U\rho} = dS.\alpha \log U\frac{\rho}{\beta}$$

which is immediately integrable, β being an arbitrary but constant vector.

As we have not introduced into this work the *logarithms* of versors, nor the corresponding *angles* of quaternions, we must refer to Hamilton's *Elements* for a farther development of this point.]

320. *To find the orthogonal trajectories of a given series of surfaces.*

If the equation

$$F\rho = C,$$

give $$S\nu d\rho = 0,$$

the equations of the orthogonal curves is

$$V\nu d\rho = 0.$$

This is equivalent to two scalar differential equations (§ 197), which, when the problem is possible, belong to surfaces on each of which the required lines lie. The finding of the requisite criterion we leave to the student.

Let the surfaces be concentric spheres.

Here $$\rho^2 = C,$$

and therefore $$V\rho d\rho = 0.$$

Hence $$dU\rho = -U\rho V\rho d\rho = 0,$$

and the integral is

$$U\rho = \text{constant},$$

denoting straight lines through the origin.

Let the surfaces be spheres touching each other at a common point. The equation is (§ 218)

$$S\alpha\rho^{-1} = C,$$

whence $$V.\rho\alpha\rho d\rho = 0.$$

The integrals may be written

$$S.\alpha\beta\rho = 0, \qquad \rho^2 + kTV\alpha\rho = 0,$$

the first (β being any vector) is a plane through the common diameter; the second represents a series of rings or *tores* (§ 322) formed by the revolution, about a, of circles *touching* that line at the point common to the spheres.

Let the surfaces be similar, similarly situated, and concentric, surfaces of the second order.

Here $$S\rho\chi\rho = C,$$
therefore $$V\chi\rho d\rho = 0.$$

But, by § 290, the integral of this equation is

$$\begin{aligned}\rho &= e^{t\chi}\epsilon \\ &= \phi^t\epsilon,\end{aligned}$$

where ϕ and χ are related to each other, as in § 290; and ϵ is any constant vector.

321. *Find the general equation of surfaces described by a line which always meets, at right angles, a fixed line.*

If a be the fixed line, β and γ forming with it a rectangular unit system, then

$$\rho = xa + y(\beta + z\gamma),$$

where y may have all values, but x and z are mutually dependent, is one form of the equation.

Another, expressing the arbitrary relation between x and z, is

$$\frac{S\gamma\rho}{S\beta\rho} = f(Sa\rho).$$

But we may also write

$$\rho = aF(x) + ya^x\beta,$$

as it obviously expresses the same conditions.

The simplest case is when $F(x) = kx$. The surface is one which cuts, in a right helix, every cylinder which has a for its axis.

322. *The centre of a sphere moves in a given circle, find the equation of the ring described.*

Let a be the unit-vector axis of the circle, its centre the origin, r its radius, a that of the sphere.

Then $$(\rho-\beta)^2 = -a^2$$
is the equation of the sphere in any position, where
$$S\alpha\beta = 0, \qquad T\beta = r.$$
These give $S.\alpha\beta\rho = 0$, and β must now be eliminated. The result is that
$$\beta = r\alpha UV\alpha\rho,$$
giving
$$(\rho^2-r^2+a^2)^2 = 4r^2T^2V\alpha\rho,$$
$$= 4r^2(-\rho^2-S^2\alpha\rho),$$
which is the required equation. It may easily be changed to
$$(\rho^2-a^2+r^2)^2 = -4a^2\rho^2-4r^2S^2\alpha\rho, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$
and in this form it enables us to give an immediate proof of the very singular property of the ring (or *tore*) discovered by Villarceau.

For the planes
$$S.\rho\left(\alpha \pm \frac{a\beta}{r\sqrt{r^2-a^2}}\right) = 0,$$
which together are represented by
$$r^2(r^2-a^2)S^2\alpha\rho - a^2S^2\beta\rho = 0,$$
evidently pass through the origin and touch (and cut) the ring.

The latter equation may be written
$$r^2S^2\alpha\rho - a^2(S^2\alpha\rho + S^2\rho U\beta) = 0,$$
or
$$r^2S^2\alpha\rho + a^2(\rho^2 + S^2.\alpha\rho U\beta) = 0. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

The plane intersections of (1) and (2) lie obviously on the new surface
$$(\rho^2-a^2+r^2)^2 = 4a^2S^2.\alpha\rho U\beta,$$
which consists of two spheres of radius r, as we see by writing its separate factors in the form
$$(\rho \pm a\alpha U\beta)^2 + r^2 = 0.$$

323. It may be instructive to work out this problem from a different point of view, especially as it affords excellent practice in transformations.

A circle revolves about an axis passing within it, the perpendicular from the centre on the axis lying in the plane of the circle: show that, for a certain position of the axis, the same solid may be traced out by a circle revolving about an external axis in its own plane.

Let $a = \sqrt{b^2+c^2}$ be the radius of the circle, i the vector axis of rotation, $-c\alpha$ (where $T\alpha = 1$) the vector perpendicular from the centre on the axis i, and let the vector

$$bi + ci\alpha$$

be perpendicular to the plane of the circle.

The equations of the circle are

$$\left.\begin{aligned}(\rho - c\alpha)^2 + b^2 + c^2 &= 0,\\ S\left(i + \frac{c}{b}i\alpha\right)\rho &= 0.\end{aligned}\right\}$$

Also
$$\begin{aligned}-\rho^2 &= S^2 i\rho + S^2\alpha\rho + S^2.i\alpha\rho,\\ &= S^2 i\rho + S^2\alpha\rho + \frac{b^2}{c^2}S^2 i\rho\end{aligned}$$

by the second of the equations of the circle. But, by the first,

$$(\rho^2 + b^2)^2 = 4c^2S^2\alpha\rho = -4(c^2\rho^2 + a^2S^2 i\rho),$$

which is easily transformed into

$$(\rho^2 - b^2)^2 = -4a^2(\rho^2 + S^2 i\rho),$$

or
$$\rho^2 - b^2 = -2aTVi\rho.$$

If we put this in the forms

$$\rho^2 - b^2 = 2aS\beta\rho,$$

and
$$(\rho - a\beta)^2 + c^2 = 0,$$

where β is a unit-vector perpendicular to i and in the plane of i and ρ, we see at once that the surface will be traced out by a circle of radius c, revolving about i, an axis in its own plane, distant a from its centre.

This problem is not well adapted to show the gain in brevity and distinctness which generally follows the use of quaternions;

as, from its very nature, it hints at the adoption of rectangular axes and scalar equations for its treatment, so that the solution we have given is but little different from an ordinary Cartesian one.

324. *A surface is generated by a straight line which intersects two fixed lines: find the general equation.*

If the given lines intersect, there is no surface but the plane containing them.

Let then their equations be,

$$\rho = \alpha + x\beta, \qquad \rho = \alpha_1 + x_1\beta_1.$$

Hence every point of the surface satisfies the condition, § 30,

$$\rho = y(\alpha + x\beta) + (1-y)(\alpha_1 + x_1\beta_1). \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Obviously y may have any value whatever: but to specify a particular surface we must have a relation between x and x_1. By the help of this, x_1 may be eliminated from (1), which then takes the usual form

$$\rho = \phi(x, y).$$

Or we may operate on (1) by $V.(\alpha + x\beta - \alpha_1 - x_1\beta_1)$, so that we get a vector equation equivalent to *two* scalar equations (§ 98), and not containing y. From this x and x_1 may easily be found in terms of ρ, and the general equation of the possible surfaces may be written

$$f(x, x_1) = 0,$$

where f is an arbitrary scalar function, and the values of x and x_1 are expressed in terms of ρ.

This process is obviously applicable if we have, instead of two straight lines, any two given curves through which the line must pass; and even when the tracing line is itself a given curve, situated in a given manner. But an example or two will make the whole process clear.

325. *Suppose the moveable line to be restricted by the condition that it is always parallel to a fixed plane.*

Then, in addition to (1), we have the condition

$$S\gamma(\alpha_1+x_1\beta_1-\alpha-x\beta) = 0,$$

γ being a vector perpendicular to the fixed plane.

We lose no generality by assuming α and α_1, which are any vectors drawn from the origin to the fixed lines, to be each perpendicular to γ; for, if for instance we could not assume $S\gamma\alpha = 0$, it would follow that $S\gamma\beta = 0$, and the required surface would either be impossible, or would be a plane, cases which we need not consider. Hence

$$x_1 S\gamma\beta_1 - x S\gamma\beta = 0.$$

Eliminating x_1, by the help of this equation, from (1) of last section, we have

$$\rho = y(\alpha+x\beta)+(1-y)\left(\alpha_1+x\beta_1\frac{S\gamma\beta}{S\gamma\beta_1}\right).$$

Operating by any three non-coplanar vectors, we obtain three equations from which to eliminate x and y. Operating by $S.\gamma$ we find

$$S\gamma\rho = xS\beta\gamma.$$

Eliminating x by means of this, we have finally

$$S.\rho\left(\alpha+\frac{\beta S\gamma\rho}{S\beta\gamma}\right)\left(\alpha_1+\frac{\beta_1 S\gamma\rho}{S\beta_1\gamma}\right) = 0,$$

which appears to be of the third order. It is really, however, only of the second order, since, consistently with our assumptions, we have

$$V\alpha\alpha_1 \parallel \gamma,$$

and therefore $S\gamma\rho$ is a spurious factor of the left-hand side.

326. *Let the fixed lines be perpendicular to each other, and let the moveable line pass through the circumference of a circle, whose centre is in the common perpendicular, and whose plane bisects that line at right angles.*

Here the equations of the fixed lines may be written

$$\rho = \alpha+x\beta, \qquad \rho = -\alpha+x_1\gamma,$$

where α, β, γ form a rectangular system, and we may assume the two latter to be unit-vectors.

The circle has the equations

$$\rho^2 = -\alpha^2, \qquad S\alpha\rho = 0.$$

Equation (1) of § 324 becomes

$$\rho = y(\alpha + x\beta) + (1-y)(-\alpha + x_1\gamma).$$

Hence $\quad S\alpha^{-1}\rho = y-(1-y) = 0, \quad \text{or} \quad y = \frac{1}{2}.$

Also $\quad \rho^2 = -\alpha^2 = (2y-1)^2\alpha^2 - x^2y^2 - x_1^2(1-y)^2,$

$$\text{or} \qquad 4\alpha^2 = (x^2 + x_1^2),$$

so that if we now suppose the tensors of β and γ to be each 2α, we may put $x = \cos\theta$, $x_1 = \sin\theta$, from which

$$\rho = (2y-1)\alpha + y\beta\cos\theta + (1-y)\gamma\sin\theta;$$

and finally $\quad \dfrac{S^2\beta\rho}{(1+S\alpha^{-1}\rho)^2} + \dfrac{S^2\gamma\rho}{(1-S\alpha^{-1}\rho)^2} = 4\alpha^4.$

For this very simple case the solution is not better than the ordinary Cartesian one; but the student will easily see that we may by very slight changes adapt the above to data far less symmetrical than those from which we started. Suppose, for instance, β and γ not to be at right angles to one another; and suppose the plane of the circle not to be parallel to their plane, &c., &c. But farther, operate on every line in space by the linear and vector function ϕ, and we distort the circle into an ellipse, the straight lines remaining straight. If we choose a form of ϕ whose principal axes are parallel to α, β, γ, the data will remain symmetrical, but not unless. This subject will be considered again in the next Chapter.

327. *To find the curvature of a normal section of a central surface of the second order.*

In this, and the few similar investigations which follow, it will be simpler to employ infinitesimals than differentials; though for a thorough treatment of the subject the latter method, as may be seen in Hamilton's *Elements*, is preferable.

We have, of course,

$$S\rho\phi\rho = 1,$$

and, if $\rho+\delta\rho$ be also a vector of the surface, we have rigorously, *whatever be the tensor of* $\delta\rho$,

$$S(\rho+\delta\rho)\phi(\rho+\delta\rho) = 1.$$

Hence $$2S\delta\rho\phi\rho + S\delta\rho\phi\delta\rho = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Now $\phi\rho$ is normal to the tangent plane at the extremity of ρ, so that if t denote the distance of the point $\rho+\delta\rho$ from that plane

$$t = S\delta\rho U\phi\rho,$$

and (1) may therefore be written

$$2tT\phi\rho - T^2\delta\rho S.U\delta\rho\phi U\delta\rho = 0.$$

But the curvature of the section is evidently

$$\mathcal{L}\frac{2t}{T^2\delta\rho},$$

or, by the last equation,

$$\frac{1}{T\phi\rho}\mathcal{L}S.U\delta\rho\phi U\delta\rho.$$

In the limit, $\delta\rho$ is a vector in the tangent plane; let ϖ be the vector semidiameter of the surface which is parallel to it, and the equation of the surface gives

$$T^2\varpi S.U\varpi\phi U\varpi = 1,$$

so that the curvature of the normal section, at the point ρ, in the direction of ϖ, is

$$\frac{1}{T\phi\rho T^2\varpi},$$

directly as the perpendicular from the centre on the tangent plane, and inversely as the square of the semidiameter parallel to the tangent line, a well-known theorem.

328. By the help of the known properties of the central section parallel to the tangent plane, this theorem gives us all the ordinary properties of the directions of maximum and minimum curvature, their being at right angles to each other, the curvature in any normal section in terms of the chief curvatures and the inclination to their planes, &c., &c., without farther analysis. And when, in a future section, we show how to find an *osculating* surface of the second order at any point of a given surface, the same properties will be at once established for surfaces in general. Meanwhile we may prove another curious property of the surfaces of the second order, which similar reasoning extends to all surfaces.

The equation of the normal at the point $\rho+\delta\rho$ in the surface treated in last section is

$$\varpi = \rho+\delta\rho+x\phi(\rho+\delta\rho). \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This intersects the normal at ρ if (§§ 203, 210)

$$S.\delta\rho\phi\rho\phi\delta\rho = 0,$$

that is, by the result of § 273, if $\delta\rho$ be parallel to the maximum or minimum diameter of the central section parallel to the tangent plane.

Let σ_1 and σ_2 be those diameters, then we may write in general

$$\delta\rho = p\sigma_1+q\sigma_2,$$

where p and q are scalars, infinitely small.

If we draw through a point P in the normal at ρ a line parallel to σ_1, we may write its equation

$$\varpi = \rho+a\phi\rho+y\sigma_1.$$

The proximate normal (1) passes this line at a distance (see § 203)

$$S.(a\phi\rho-\delta\rho)UV\sigma_1\phi(\rho+\delta\rho),$$

or, neglecting terms of the second order,

$$\frac{1}{TV\sigma_1\phi\rho}(apS.\phi\rho\sigma_1\phi\sigma_1+aqS.\phi\rho\sigma_1\phi\sigma_2+qS.\sigma_1\sigma_2\phi\rho).$$

The first term in the bracket vanishes because σ_1 is a principal vector of the section parallel to the tangent plane, and thus the expression becomes

$$q\left(\frac{a}{T\sigma_2} - T\sigma_2\right).$$

Hence, if we take $a = T\sigma_2^2$, the distance of the normal from the new line is of the second order only. This makes the distance of P from the point of contact $T\phi\rho\, T\sigma_2^2$, i. e. the principal radius of curvature along the tangent line parallel to σ_2. That is *the group of normals drawn near a point of a central surface of the second order pass ultimately through two lines each parallel to the tangent to one principal section, and passing through the centre of curvature of the other.* The student may form a notion of the nature of this proposition by considering a small square plate, with normals drawn at every point, to be slightly bent, but by different amounts, in planes perpendicular to its edges. The first bending will make all the normals pass through the axis of the cylinder of which the plate now forms part; the second bending will not sensibly disturb this arrangement, except by lengthening or shortening the line in which the normals meet, but it will make them meet also in the axis of the new cylinder, at right angles to the first. A small pencil of light, with its focal lines, presents a similar appearance.

329. To extend these theorems to surfaces in general, it is only necessary, as Hamilton has shown, to prove that if we write

$$d\nu = \phi\, d\rho,$$

ϕ is a *self-conjugate* function; and then the properties of ϕ, as explained in preceding Chapters, are applicable to the question.

As the reader will easily see, this is merely another form of the investigation contained in § 317. But it is given here to show what a number of very simple demonstrations may be given of almost all quaternion theorems.

Now ν is defined by an equation of the form

$$df\rho = S\nu d\rho,$$

where f is a scalar function. Operating on this by another independent symbol of differentiation, δ, we have

$$\delta d f\rho = S\delta\nu d\rho + S\nu\delta d\rho.$$

In the same way we have

$$d\delta f\rho = Sd\nu\delta\rho + S\nu d\delta\rho.$$

But, as d and δ are independent, the left-hand members of these equations, as well as the second terms on the right (if these exist at all), are equal, so that we have

$$Sd\nu\delta\rho = S\delta\nu d\rho.$$

This becomes, putting $d\nu = \phi d\rho$,

and therefore $\delta\nu = \phi\delta\rho$,

$$S\delta\rho\phi d\rho = Sd\rho\phi\delta\rho,$$

which proves the proposition.

330. If we write the differential of the equation of a surface in the form

$$df\rho = 2S\nu d\rho,$$

then it is easy to see that

$$f(\rho + d\rho) = f\rho + 2S\nu d\rho + Sd\nu d\rho + \&c.,$$

the remaining terms containing as factors the third and higher powers of $Td\rho$. To the second order, then, we may write, except for certain singular points,

$$0 = 2S\nu d\rho + Sd\nu d\rho,$$

and, as before, (§ 327), the curvature of the normal section whose tangent line is $d\rho$ is

$$\frac{1}{T\nu} S\frac{d\nu}{d\rho}.$$

331. The step taken in last section, although a very simple one, virtually implies that the first three terms of the expansion of $f(\rho + d\rho)$ are to be formed in accordance with Taylor's Theorem, whose applicability to the expansion of scalar functions of quaternions has not been proved in this work, (see § 135);

we therefore give another investigation of the curvature of a normal section, employing for that purpose the formulae of § (282).

We have, treating $d\rho$ as an element of a curve,

$$S\nu d\rho = 0,$$

or, making s the independent variable,

$$S\nu\rho' = 0.$$

From this, by a second differentiation,

$$S\frac{d\nu}{ds}\rho' + S\nu\rho'' = 0.$$

The curvature is, therefore, since $\nu \| \rho''$ and $T\rho' = 1$,

$$T\rho'' = -\frac{1}{T\nu}S\frac{d\nu}{d\rho}\rho'^2 = \frac{1}{T\nu}S\frac{d\nu}{d\rho}, \text{ as before.}$$

332. Since we have shown that

$$d\nu = \phi\, d\rho$$

where ϕ is a self-conjugate linear and vector function, whose constants depend only upon the nature of the surface, and the position of the point of contact of the tangent plane; so long as we do not alter these we must consider ϕ as possessing the properties explained in Chapter V.

Hence, as the expression for $T\rho''$ does not involve the tensor of $d\rho$, we may put for $d\rho$ any unit-vector τ, subject of course to the condition

$$S\nu\tau = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

And the curvature of the normal section whose tangent is τ is

$$\frac{1}{T\nu}S\frac{\phi\tau}{\tau} = -\frac{1}{T\nu}S\tau\phi\tau.$$

If we consider the central section of the surface of the second order

$$S\varpi\phi\varpi + T\nu = 0,$$

made by the plane

$$S\nu\varpi = 0,$$

we see at once that the *curvature of the given surface along the* *normal section touched by* τ *is inversely as the square of the parallel radius in the auxiliary surface.* This, of course, includes Euler's and other well-known Theorems.

333. *To find the directions of maximum and minimum curvature,* we have

$$S\tau\phi\tau = \text{max. or min.}$$

with the conditions,

$$S\nu\tau = 0,$$

$$T\tau = 1.$$

By differentiation, as in § 273, we obtain the farther equation

$$S.\nu\tau\phi\tau' = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

If τ be one of the two required directions $\tau' = \tau U\nu$ is the other, for the last-written equation may be put in the form

$$S.\tau U\nu\phi(\nu\tau U\nu) = 0,$$

$$\text{i.e.} \qquad S.\tau'\phi(\nu\tau') = 0,$$

$$\text{or} \qquad S.\nu\tau'\phi\tau' = 0.$$

Hence the sections of greatest and least curvature are perpendicular to one another.

We easily obtain, as in § 273, the following equation

$$S.\nu(\phi + S\tau\phi\tau)^{-1}\nu = 0,$$

whose roots divided by $T\nu$ are the required curvatures.

334. Before leaving this very brief introduction to a subject, an exhaustive treatment of which will be found in Hamilton's *Elements*, we may make a remark on equation (1) of last section

$$S.\nu\tau\phi\tau = 0,$$

or, as it may be written, by returning to the notation of § 332,

$$S.\nu\, d\rho\, d\nu = 0.$$

This is the general equation of lines of curvature. For, if we define a line of curvature on any surface as a line such that normals drawn at contiguous points in it intersect, then, $\delta\rho$ being an element of such a line, the normals

$$\varpi = \rho + x\nu \quad \text{and} \quad \varpi = \rho + \delta\rho + y(\nu + \delta\nu)$$

must intersect. This gives, by § 203, the condition

$$S.\delta\rho\nu\delta\nu = 0,$$

as above.

EXAMPLES TO CHAPTER IX.

1. Find the length of any arc of a curve drawn on a sphere so as to make a constant angle with a fixed diameter.

2. Show that, if the normal plane of a curve always contains a fixed line, the curve is a circle.

3. Find the radius of spherical curvature of the curve

$$\rho = \phi t.$$

Also find the equation of the locus of the centre of spherical curvature.

4. (Hamilton, *Bishop Law's Premium Examination*, 1854.)

(*a*.) If ρ be the variable vector of a curve in space, and if the differential $d\kappa$ be treated as $= 0$, then the equation

$$dT(\rho - \kappa) = 0$$

obliges κ to be the vector of some point in the normal plane to the curve.

(*b.*) In like manner the system of two equations, where $d\kappa$ and $d^2\kappa$ are each $= 0$,

$$dT(\rho-\kappa) = 0, \qquad d^2T(\rho-\kappa) = 0,$$

represents the axis of the element, or the right line drawn through the centre of the osculating circle, perpendicular to the osculating plane.

(*c.*) The system of the three equations, in which κ is treated as constant,

$$dT(\rho-\kappa) = 0, \qquad d^2T(\rho-\kappa) = 0, \qquad d^3T(\rho-\kappa) = 0,$$

determines the vector κ of the centre of the osculating sphere.

(*d.*) For the three last equations we may substitute the following:

$$S.(\rho-\kappa)\,d\rho = 0,$$
$$S.(\rho-\kappa)d^2\rho + d\rho^2 = 0,$$
$$S.(\rho-\kappa)d^3\rho + 3\,S.d\rho d^2\rho = 0.$$

(*e.*) Hence, generally, whatever the independent and scalar variable may be, on which the variable vector ρ of the curve depends, the vector κ of the centre of the osculating sphere admits of being thus expressed:

$$\kappa = \rho + \frac{3\,V.d\rho d^2\rho\, S.d\rho d^2\rho - d\rho^2\, V.d\rho d^3\rho}{S.d\rho d^2\rho d^3\rho}.$$

(*f.*) In general,

$$d(d^{-2}V.d\rho U\rho) = d(T\rho^{-3}V.\rho d\rho)$$
$$= T\rho^{-5}(3\,V.\rho d\rho\, S.\rho d\rho - \rho^2 V.\rho d^2\rho);$$

whence,

$$3\,V.\rho d\rho S.\rho d\rho - \rho^2 V.\rho d^2\rho = \rho^4\, T\rho d(\rho^{-2}V.d\rho U\rho);$$

and, therefore, the recent expression for κ admits of being thus transformed,

$$\kappa = \rho + \frac{d\rho^4 d(d\rho^{-2}V.d^2\rho U d\rho)}{S.d^2\rho d^3\rho U d\rho}.$$

(*g.*) If the length of the element of the curve be constant, $dTd\rho = 0$, this last expression for the vector of the centre of the osculating sphere to a curve of double curvature becomes, more simply,

$$\kappa = \rho + \frac{d.d^2\rho d\rho^3}{S.d\rho d^2\rho d^3\rho};$$

or

$$\kappa = \rho + \frac{V.d^3\rho d\rho^2}{S.d\rho d^2\rho d^3\rho}.$$

(*h.*) Verify that this expression gives $\kappa = 0$, for a curve described on a sphere which has its centre at the origin of vectors; or show that whenever $dT\rho = 0$, $d^2T\rho = 0$, $d^3T\rho = 0$, as well as $dTd\rho = 0$, then

$$\rho S.d\rho^{-1} d^2\rho d^3\rho = V.d\rho d^3\rho.$$

5. Find the curve from every point of which three given spheres will appear of equal magnitude.

6. Show that the locus of a point, the difference of whose distances from each two of three given points is constant, is a plane curve.

7. Find the equation of the curve which cuts at a given angle all the sides of a cone of the second order.

Find the length of any arc in terms of the distances of its extremities from the vertex.

8. Why is the centre of spherical curvature, of a curve described on a sphere, not necessarily the centre of the sphere?

9. Find the equation of the developable surface whose generating lines are the intersections of successive normal planes to a given tortuous curve.

10. Find the length of an arc of a tortuous curve whose normal planes are equidistant from the origin.

11. The reciprocals of the perpendiculars from the origin on the tangent planes to a developable surface are vectors of a tortuous curve; from whose osculating planes the cusp-edge of the original surface may be reproduced by the same process.

12. The equation

$$\rho = Va^t \beta,$$

where a is a unit-vector not perpendicular to β, represents an ellipse. If we put $\gamma = Va\beta$, show that the equations of the locus of the centre of curvature are

$$S.\beta\gamma\rho = 0,$$

$$S^{\frac{2}{3}}\beta\rho + S^{\frac{2}{3}}\gamma\rho = (\beta SUa\beta)^{\frac{4}{3}}.$$

13. Find the radius of absolute curvature of a spherical conic.

14. If a cone be cut in a circle by a plane perpendicular to a side, the axis of the right cone which osculates it, along that side, passes through the centre of the section.

15. Show how to find the vector of an umbilicus. Apply your method to the surfaces whose equations are

$$S\rho\phi\rho = 1,$$

and $$Sa\rho S\beta\rho S\gamma\rho = 1.$$

16. Find the locus of the umbilici of the surfaces represented by the equation

$$S\rho(\phi + h)^{-1}\rho = 1,$$

where h is an arbitrary parameter.

17. Show how to find the equation of a tangent plane which touches a surface along a line. Find such planes for the following surfaces

$$S\rho\phi\rho = 1,$$

$$S\rho(\phi - \rho^2)^{-1}\rho = 1,$$

and $$(\rho^2 - a^2 + b^2)^2 + 4(a^2\rho^2 + b^2 S^2 a\rho) = 0.$$

18. Find the condition that the equation

$$S(\rho + a)\phi\rho = 1,$$

where ϕ is a self-conjugate linear and vector function, may represent a cone.

19. Show from the general equation that cones and cylinders are the only developable surfaces of the second order.

20. Find the equation of the envelop of planes drawn at each point of an ellipsoid perpendicular to the radius vector from the centre.

21. Find the equation of the envelop of spheres whose centres lie on a given sphere, and which pass through a given point.

22. Find the locus of the foot of the perpendicular from the centre to the tangent plane of a hyperboloid of one, or of two, sheets.

23. Hamilton, *Bishop Law's Premium Examination*, 1852.

(*a.*) If ρ be the vector of a curve in space, the length of the element of that curve is $Td\rho$; and the variation of the length of a finite arc of the curve is

$$\delta\int Td\rho = -\int SUd\rho\,\delta d\rho = -\Delta SUd\rho\,\delta\rho + \int SdUd\rho\,\delta\rho.$$

(*b.*) Hence, if the curve be a shortest line on a given surface, for which the normal vector is ν, so that $S\nu\delta\rho = 0$, this shortest or geodetic curve must satisfy the differential equation,

$$V\nu dUd\rho = 0.$$

Also, for the extremities of the arc, we have the limiting equations,

$$SUd\rho_0\,\delta\rho_0 = 0; \qquad SUd\rho_1\,\delta\rho_1 = 0.$$

Interpret these results.

(*c.*) For a spheric surface, $V\nu\rho = 0$, $\rho dUd\rho = 0$; the integrated equation of the geodetics is $\rho Ud\rho = \varpi$, giving $S\varpi\rho = 0$ (great circle).

For an arbitrary cylindric surface,

$$S\alpha\nu = 0, \qquad \alpha dUd\rho = 0;$$

the integral shows that the geodetic is generally a helix, making a constant angle with the generating lines of the cylinder.

(*d.*) For an arbitrary conic surface,

$$S\nu\rho = 0, \qquad S\rho dUd\rho = 0;$$

integrate this differential equation, so as to deduce from it, $TV\rho Ud\rho =$ const.

Interpret this result; shew that the perpendicular from the vertex of the cone on the tangent to a given geodetic line is constant; this gives the rectilinear development.

When the cone is of the second degree, the same property is a particular case of a theorem respecting confocal surfaces.

(*e.*) For a surface of revolution,

$$S.\alpha\rho\nu = 0, \qquad S.\alpha\rho dUd\rho = 0;$$

integration gives,

$$\text{const.} = S.\alpha\rho Ud\rho = TV\alpha\rho SU(V\alpha\rho.d\rho);$$

the perpendicular distance of a point on a geodetic line from the axis of revolution varies inversely as the cosine of the angle under which the geodetic crosses a parallel (or circle) on the surface.

(*f.*) The differential equation, $S.\alpha\rho dUd\rho = 0$, is satisfied not only by the geodetics, but also by the circles, on a surface of revolution; give the explanation of this fact of calculation, and show that it arises from the coincidence between the normal plane to the circle and the plane of the meridian of the surface.

(*g.*) For any arbitrary surface, the equation of the geodetic may be thus transformed, $S.\nu d\rho d^2\rho = 0$; deduce this form, and show that it expresses the normal property of the osculating plane.

(*h.*) If the element of the geodetic be constant, $dTd\rho = 0$, then the general equation formerly assigned may be reduced to $V.\nu d^2\rho = 0$.

Under the same condition, $d^2\rho = -\nu^{-1}Sd\nu d\rho$.

(*i.*) If the equation of a central surface of the second order be put under the form $f\rho = 1$, where the function f is scalar, and homogeneous of the second dimension, then the differential of that function is of the form $df\rho = 2S.\nu d\rho$, where the normal vector, $\nu = \phi\rho$, is a distributive function of ρ (homogeneous of the first dimension), $d\nu = d\phi\rho = \phi d\rho$.

This normal vector ν may be called the *vector of proximity* (namely, of the element of the surface to the centre); because its reciprocal, ν^{-1}, represents in length and in direction the perpendicular let fall from the centre on the tangent plane to the surface.

(*k.*) If we make $S\sigma\phi\rho = f(\sigma, \rho)$, *this* function f is commutative with respect to the *two* vectors on which it depends, $f(\rho, \sigma) = f(\sigma, \rho)$; it is also connected with the *former* function f, of a *single* vector ρ, by the relation, $f(\rho, \rho) = f\rho$: so that $f\rho = S\rho\phi\rho$.

$fd\rho = Sd\rho d\nu$; $dfd\rho = 2S.d\nu d^2\rho$; for a geodetic, with constant element,

$$\frac{dfd\rho}{2fd\rho} + S\frac{d\nu}{\nu} = 0;$$

this equation is immediately integrable, and gives const. $= T\nu\sqrt{(fUd\rho)}$ = reciprocal of Joachimstal's product, *PD*.

(*l.*) If we give the name of "Didonia" to the curve (discussed by Delaunay) which, on a given surface and with a given perimeter, contains the greatest area, then for such a Didonian curve we have by quaternions the formula,

$$\int S.U\nu d\rho\delta\rho + c\delta\int Td\rho = 0,$$

where c is an arbitrary constant.

Derive hence the differential equation of the second order, equivalent (through the constant c) to one of the third order,

$$c^{-1}d\rho = V.U\nu dUd\rho.$$

Geodetics are, therefore, that limiting case of Didonias for which the constant c is infinite.

On a plane, the Didonia is a circle, of which the equation, obtained by integration from the general form, is

$$\rho = \varpi + cU\nu d\rho,$$

ϖ being vector of centre, and c being radius of circle.

(*m.*) Operating by $S.Ud\rho$, the general differential equation of the Didonia takes easily the following forms:

$$c^{-1}Td\rho = S(U\nu d\rho . dUd\rho);$$

$$c^{-1}Td\rho^2 = S(U\nu d\rho . d^2\rho);$$

$$c^{-1}Td\rho^3 = S.U\nu d\rho d^2\rho;$$

$$c^{-1} = S\frac{d^2\rho\, d\rho^{-2}}{U\nu d\rho}.$$

(*n.*) The vector ω, of the centre of the osculating circle to a curve in space, of which the element $Td\rho$ is constant, has for expression,

$$\omega = \rho + \frac{d\rho^2}{d^2\rho}.$$

Hence for the general Didonia,

$$c^{-1} = S\frac{(\omega-\rho)^{-1}}{U\nu d\rho};$$

$$T(\rho-\omega) = cSU\frac{\rho-\omega}{\nu d\rho}.$$

(*o.*) Hence, the radius of curvature of any one Didonia varies, in general, proportionally to the cosine of the inclination of the osculating plane of the curve to the tangent plane of the surface.

And hence, by Meusnier's theorem, the difference of the squares of the curvatures of curve and surface is constant; the curvature of the surface meaning here the reciprocal of the radius of the sphere which osculates in the reduction of the element of the Didonia.

(*p*.) In general, for any curve on any surface, if ξ denote the vector of the intersection of the axis of the element (or the axis of the circle osculating to the curve) with the tangent plane to the surface, then

$$\xi = \rho + \frac{\nu d\rho^3}{S.\nu d\rho d^2\rho}.$$

Hence, for the general Didonia, with the same signification of the symbols,

$$\xi = \rho - cU\nu d\rho;$$

and the constant c expresses the length of the interval $\rho-\xi$, intercepted on the tangent plane, between the point of the curve and the axis of the osculating circle.

(*q*.) If, then, a sphere be described, which shall have its centre on the tangent plane, and shall contain the osculating circle, the radius of this sphere shall always be equal to c.

(*r*.) The recent expression for ξ, combined with the first form of the general differential equation of the Didonia, gives

$$d\xi = -cVdU\nu Ud\rho; \qquad V\nu d\xi = 0.$$

(*s*.) Hence, or from the geometrical signification of the constant c, the known property may be proved, that if a developable surface be circumscribed about the arbitrary surface, so as to touch it along a Didonia, and if this developable be then unfolded into a plane, the curve will at the same time be flattened (generally) into a circular arc, with radius $= c$.

24. Find the condition that the equation

$$S\rho(\phi+f)^{-1}\rho = 1$$

may give three real values of f for any given value of ρ. If f be a function of a scalar parameter ξ, show how to find the form of this function in order that we may have

$$-\nabla^2\xi = \frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} + \frac{d^2\xi}{dz^2} = 0.$$

Prove that the following is the relation between f and ξ,

$$c\xi = \int \frac{df}{\sqrt{(g_1+f)(g_2+f)(g_3+f)}} = \int \frac{df}{\sqrt{m_f}}$$

in the notation of § 147.

25. Show, after Hamilton, that the proof of Dupin's theorem, that "each member of one of three series of orthogonal surfaces cuts each of the other series along its lines of curvature," may be expressed in quaternion notation as follows:

If $\quad S\nu d\rho = 0, \quad S\nu' d\rho = 0, \quad S.\nu\nu' d\rho = 0$

be integrable, and if

$S\nu\nu' = 0, \quad$ then $\quad V\nu' d\rho = 0 \quad$ makes $\quad S.\nu\nu' d\nu = 0.$

Or, as follows,

If $\quad S\nu\nabla\nu = 0, \quad S\nu'\nabla\nu' = 0, \quad S\nu''\nabla\nu'' = 0, \quad$ and $\quad V.\nu\nu'\nu'' = 0,$

then $\quad S.\nu''(S\nu'\nabla.\nu) = 0,$

where $\quad \nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$

26. Show that the equation

$$V\alpha\rho = \rho V\beta\rho$$

represents the line of intersection of a cylinder and cone, of the second order, which have β as a common generating line.

CHAPTER X.

KINEMATICS.

335. When a point's vector, ρ, is a function of the time t, we have seen (§ 35) that its vector-velocity is expressed by $\frac{d\rho}{dt}$ or, in Newton's notation, by $\dot{\rho}$.

That is, if
$$\rho = \phi t$$
be the equation of an orbit, *containing* (as the reader may see) *not merely the form of the orbit, but the law of its description also,* then
$$\dot{\rho} = \phi' t$$
gives at once the form of the *Hodograph* and the law of its description.

This shows immediately that the *vector-acceleration of a point's motion*
$$\frac{d^2\rho}{dt^2} \text{ or } \ddot{\rho},$$
is the vector-velocity in the hodograph. Thus the fundamental properties of the hodograph are proved almost intuitively.

336. Changing the independent variable, we have
$$\dot{\rho} = \frac{d\rho}{ds}\frac{ds}{dt} = v\rho',$$
if we employ the dash, as before, to denote $\frac{d}{ds}$.

This merely shows, in another form, that $\dot{\rho}$ expresses the velocity in magnitude and direction. But a second differentiation gives
$$\ddot{\rho} = \dot{v}\rho' + v^2\rho''.$$
This shows that the vector-acceleration can be resolved into two components, the first, $\dot{v}\rho'$, being in the direction of motion and

equal in magnitude to the acceleration of the velocity, $\dot{v}$ or $\frac{dv}{dt}$; the second, $v^2\rho''$, being in the direction of the radius of absolute curvature, and its amount equal to the square of the velocity multiplied by the curvature.

[It is scarcely conceivable that this important fundamental proposition, of which no simple analytical proof seems to have been obtained by Cartesian methods, can be proved more elegantly than by the process just given.]

337. If the motion be in a plane curve, we may write the equation as follows, so as to introduce the usual polar coördinates, r and θ,

$$\rho = r\alpha^{\frac{2\theta}{\pi}}\beta,$$

where α is a unit vector perpendicular to, β a unit vector in, the plane of the curve.

Here, of course, r and θ may be considered as connected by one scalar equation; or better, each may be looked on as a function of t. By differentiation we get

$$\dot{\rho} = \dot{r}\alpha^{\frac{2\theta}{\pi}}\beta + r\dot{\theta}\alpha\alpha^{\frac{2\theta}{\pi}}\beta,$$

which shows at once that $\dot{r}$ is the velocity along, $r\dot{\theta}$ that perpendicular to, the radius vector. Again,

$$\ddot{\rho} = (\ddot{r} - r\dot{\theta}^2)\alpha^{\frac{2\theta}{\pi}}\beta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\alpha\alpha^{\frac{2\theta}{\pi}}\beta,$$

which gives, by inspection, the components of acceleration along, and perpendicular to, the radius vector.

338. For *uniform acceleration in a constant direction*, we have at once,

$$\ddot{\rho} = \alpha.$$

Whence $$\dot{\rho} = \alpha t + \beta,$$

where β is the vector-velocity at epoch. This shows that the hodograph is a straight line described uniformly.

Also $$\rho = \frac{\alpha t^2}{2} + \beta t,$$

no constant being added if the origin be assumed to be the position of the moving point at epoch.

Since the resolved parts of ρ, parallel to β and α, vary respectively as the first and second powers of t, the curve is evidently a parabola (§ 31 (*f*)).

But we may easily deduce from the equation the following result,

$$T(\rho + \tfrac{1}{2}\beta\alpha^{-1}\beta) = -SU\alpha\left(\rho + \frac{\beta^2}{2}\alpha^{-1}\right),$$

the equation of a paraboloid of revolution, whose axis is α. Also $$S.\alpha\beta\rho = 0,$$
and therefore the distance of any point in the path from the point $-\tfrac{1}{2}\beta\alpha^{-1}\beta$ is equal to its distance from the line whose equation is

$$\rho = -\frac{\beta^2}{2}\alpha^{-1} + x\alpha V\alpha\beta.$$

Thus we recognise the focus and directrix property.

339. That the moving point may reach a point γ we must have, for some real value of t,

$$\gamma = \frac{\alpha}{2}t^2 + \beta t.$$

Now suppose $T\beta$, the velocity of projection, to be given $= v$, and, for shortness, write ϖ for $U\beta$.

Then $$\gamma = \frac{\alpha}{2}t^2 + vt\varpi.$$

Since $$T\varpi = 1,$$

we have $$\frac{T\alpha^2 t^4}{4} - (v^2 - S\alpha\gamma)t^2 + T\gamma^2 = 0.$$

The values of t^2 are *real* if

$$(v^2 - S\alpha\gamma)^2 - T\alpha^2 T\gamma^2$$

is positive. Now, as $TaT\gamma$ is never less than $Sa\gamma$, it is evident that $v^2 - Sa\gamma$ must always be positive if the roots are possible. Hence, when they are possible, both values of t^2 are *positive*. Thus we have *four* values of t which satisfy the conditions, and it is easy to see that since, disregarding the signs, they are equal two and two, each pair refer to the same path, but *described in opposite directions* between the origin and the extremity of γ. There are therefore, if any, in general two parabolas which satisfy the conditions. The directions of projection are (of course) given by the corresponding values of ϖ.

340. The envelop of all the trajectories possible with a given velocity, evidently corresponds to

$$(v^2 - Sa\gamma)^2 - Ta^2 T\gamma^2 = 0,$$

for then γ is the vector of intersection of two indefinitely close paths in the same vertical plane.

Now $$v^2 - Sa\gamma = TaT\gamma$$

is evidently the equation of a paraboloid of revolution of which the origin is the focus, the axis parallel to a, and the directrix plane at a distance $\frac{v^2}{Ta}$.

All the ordinary problems connected with parabolic motion are easily solved by means of the above formulae. Some, however, are even more easily treated by assuming a horizontal unit vector in the plane of motion, and expressing β in terms of it and a. But this must be left to the student.

341. For *acceleration directed to or from a fixed point*, we have, taking that point as origin, and putting P for the magnitude of the central acceleration,

$$\ddot{\rho} = PU\rho.$$

Whence, at once, $$V\rho\ddot{\rho} = 0.$$

Integrating, $$V\rho\dot{\rho} = \gamma = \text{a constant vector.}$$

The interpretation of this simple formula is—*first*, ρ and $\dot{\rho}$ are in a plane perpendicular to γ, hence the path is in a plane (of course passing through the origin); *second,* the area of the triangle, two of whose sides are ρ and $\dot{\rho}$, is constant.

[It is scarcely possible to imagine that a more simple proof than this can be given of the fundamental facts, that a central orbit is a plane curve, and that equal areas are described by the radius vector in equal times.]

342. When the *law of acceleration to or from the origin is that of the inverse square of the distance,* we have

$$P = \frac{\mu}{T\rho^2},$$

where μ is *negative* if the acceleration be directed *to* the origin.

Hence $$\ddot{\rho} = \frac{\mu U\rho}{T\rho^2}.$$

The following beautiful method of integration is due to Hamilton. (See Chapter IV.)

Generally, $$\frac{dU\rho}{dt} = -\frac{U\rho . V\rho\dot{\rho}}{T\rho^2} = -\frac{U\rho . \gamma}{T\rho^2},$$

therefore $$\ddot{\rho}\gamma = -\mu\frac{dU\rho}{dt},$$

and $$\dot{\rho}\gamma = \epsilon - \mu U\rho,$$

where ϵ is a constant vector, perpendicular to γ, because

$$S\gamma\dot{\rho} = 0.$$

Hence, in this case, we have for the hodograph,

$$\dot{\rho} = \epsilon\gamma^{-1} - \mu U\rho . \gamma^{-1}.$$

Of the two parts of this expression, which are both vectors, the first is constant, and the second is constant in length. Hence the locus of the extremity of $\dot{\rho}$ is a circle, whose radius is $\frac{\mu}{T\gamma}$, and whose centre is at the extremity of the vector $\epsilon\gamma^{-1}$.

[This equation contains the whole theory of the *Circular*

Hodograph. Its consequences are developed at length in Hamilton's *Elements.*]

343. We may write the equations of this circle in the form

$$T(\dot{\rho}-\epsilon\gamma^{-1}) = \frac{\mu}{T\gamma},$$

(a sphere), and $$S\gamma\dot{\rho} = 0,$$

(a plane through the origin, and through the centre of the sphere).

The equation of the orbit is found by operating by $V.\rho$ upon that to the hodograph. We thus obtain

$$\gamma = V.\rho\epsilon\gamma^{-1}+\mu T\rho\gamma^{-1},$$

or $$\gamma^2 = S\epsilon\rho+\mu T\rho,$$

or $$\mu T\rho = S\epsilon(\gamma^2\epsilon^{-1}-\rho);$$

in which last form we at once recognise the focus and directrix property. This is in fact the equation of a conicoid of revolution about its principal axis (ϵ), and the origin is one of the foci. The orbit is found by combining it with the equation of its plane, $$S\gamma\rho = 0.$$

We see at once that $\gamma^2\epsilon^{-1}$ is the vector distance of the directrix from the focus; and similarly that the excentricity is $\frac{T\epsilon}{\mu}$, and the major axis $\frac{-2\mu\gamma^2}{\mu^2+\epsilon^2}$.

344. To take a simpler case: *let the acceleration vary as the distance from the origin.*

Then $$\ddot{\rho} = \pm m^2\rho,$$

the upper or lower sign being used according as the acceleration is *from* or *to* the centre.

This is $$\left(\frac{d^2}{dt^2}\mp m^2\right)\rho = 0.$$

Hence $$\rho = \alpha\varepsilon^{mt}+\beta\varepsilon^{-mt};$$

or
$$\rho = \alpha \cos mt + \beta \sin mt,$$

where α and β are arbitrary, but constant, vectors; and ϵ is the base of Napier's logarithms.

The first is the equation of a hyperbola of which α and β are the directions of the asymptotes; the second, that of an ellipse of which α and β are semi-conjugate diameters.

Since
$$\dot{\rho} = m\,\{\alpha\epsilon^{mt} - \beta\epsilon^{-mt}\},$$
or
$$= m\,\{-\alpha \sin mt + \beta \cos mt\},$$

the hodograph is again a hyperbola or ellipse. But in the first case it is, if we neglect the change of dimensions indicated by the scalar factor m, conjugate to the orbit; in the case of the ellipse it is similar and similarly situated.

345. Again, *let the acceleration be as the inverse third power of the distance*, we have
$$\ddot{\rho} = \frac{\mu U\rho}{T\rho^3}.$$

Of course, we have, as usual,
$$V\rho\dot{\rho} = \gamma.$$

Also, operating by $S.\dot{\rho}$,
$$S\dot{\rho}\ddot{\rho} = \frac{\mu S\rho\dot{\rho}}{T\rho^4},$$
of which the integral is
$$\dot{\rho}^2 = C - \frac{\mu}{\rho^2}$$
the equation of energy.

Again,
$$S\rho\ddot{\rho} = \frac{\mu}{\rho^2}.$$

Hence
$$S\rho\ddot{\rho} + \dot{\rho}^2 = C,$$
or
$$S\rho\dot{\rho} = Ct,$$

no constant being added if we reckon the time from the passage through the apse, where $S\rho\dot{\rho} = 0$.

We have, therefore, by a second integration,
$$\rho^2 = Ct^2 + C'. \qquad\qquad (1)$$

[To determine C', remark that

$$\rho\dot{\rho} = Ct+\gamma,$$

or

$$\rho^2\dot{\rho}^2 = C^2t^2-\gamma^2.$$

But

$$\rho^2\dot{\rho}^2 = C\rho^2-\mu \text{ (by the equation of energy)},$$

$$= C^2t^2+CC'-\mu, \text{ by (1)}.$$

Hence,

$$CC' = \mu-\gamma^2.]$$

To complete the solution, we have, by § 133,

$$V\frac{\dot{\rho}}{\rho} = \frac{dU\rho}{dt}(U\rho)^{-1} = \frac{d}{dt}\log\frac{U\rho}{\beta},$$

where β is a unit vector in the plane of the orbit.

But

$$V\frac{\dot{\rho}}{\rho} = -\frac{\gamma}{\rho^2}.$$

Hence

$$\log\frac{U\rho}{\beta} = -\gamma\int\frac{dt}{Ct^2+C'}.$$

The elimination of t between this equation and (1) gives $T\rho$ in terms of $U\rho$, or the required equation of the path.

We may remark that if θ be the ordinary polar angle in the orbit,

$$\log\frac{U\rho}{\beta} = \theta U\gamma.$$

Hence we have

$$\left.\begin{aligned} \theta &= -T\gamma\int\frac{dt}{Ct^2+C'}, \\ \text{and} \qquad r^2 &= -(Ct^2+C'), \end{aligned}\right\}$$

from which the ordinary equations of Cotes' spirals can be at once found.

346. *To find the conditions that a given curve may be the hodograph corresponding to a central orbit.*

If ϖ be its vector, *given as a function of the time*, $\int\varpi\, dt$ is that of the orbit; hence the requisite conditions are given by

$$V\varpi\int\varpi\, dt = \gamma,$$

where γ is a constant vector.

We may transform this into other shapes more resembling the Cartesian ones.

Thus $$V\dot{\varpi}\int\varpi\,dt = 0,$$

and $$V\ddot{\varpi}\int\varpi\,dt + V\dot{\varpi}\varpi = 0.$$

From the first $$\int\varpi\,dt = x\dot{\varpi},$$

and therefore $$x\,V\varpi\dot{\varpi} = \gamma,$$

or the curve is *plane*. And

$$x\,V\ddot{\varpi}\dot{\varpi} + V\dot{\varpi}\varpi = 0;$$

or eliminating x, $$\gamma\,V\dot{\varpi}\ddot{\varpi} = -(V\varpi\dot{\varpi})^2.$$

Now if v' be the velocity in the hodograph, R' its radius of curvature, p' the perpendicular on the tangent; this equation gives at once $$hv' = R'p'^2,$$

which agrees with known results.

347. *The equation of an epitrochoid or hypotrochoid,* referred to the centre of the fixed circle, is evidently

$$\rho = ai^{\frac{2\omega t}{\pi}}\alpha + bi^{\frac{2\omega_1 t}{\pi}}\alpha,$$

where α is a unit-vector in the plane of the curve and i another perpendicular to it. Here ω and ω_1 are the angular velocities in the two circles, and t is the time elapsed since the tracing point and the centres of the two circles were in one straight line.

Hence, for the length of an arc of such a curve,

$$s = \int T\dot{\rho}\,dt = \int dt\sqrt{\{\omega^2a^2 + 2\,\omega\omega_1 ab\cos(\omega-\omega_1)t + \omega_1{}^2b^2\}},$$

$$= \int dt\sqrt{\left\{(\omega a \mp \omega_1 b)^2 \pm 4\,\omega\omega_1 ab \left|\begin{array}{c}\cos^2\\ \cdot\\ \sin^2\end{array}\right| \frac{\omega-\omega_1}{2}t\right\}},$$

which is, of course, an elliptic function.

But when the curve becomes an epicycloid or a hypocycloid, $\omega a \mp \omega_1 b = 0$, and

$$s = 2\sqrt{(\pm\,\omega\omega_1 ab)}\int dt\left\{\begin{array}{c}\cos\\ \cdot\\ \sin\end{array}\right\}\frac{\omega-\omega_1}{2}t,$$

which can be expressed in finite terms, as was first shown by Newton in the *Principia.*

The hodograph is another curve of the same class, whose equation is

$$\dot{\rho} = i\left(a\omega i^{\frac{2\omega t}{\pi}}\alpha + b\omega_1 i^{\frac{2\omega_1 t}{\pi}}\alpha\right);$$

and the acceleration is denoted in magnitude and direction by the vector

$$\ddot{\rho} = -a\omega^2 i^{\frac{2\omega t}{\pi}}\alpha - b\omega_1^2 i^{\frac{2\omega_1 t}{\pi}}\alpha.$$

Of course the equations of the common *Cycloid* and *Trochoid* may be easily deduced from these forms by making a indefinitely great and ω indefinitely small, but the product $a\omega$ finite; and transferring the origin to the point

$$\rho = a\alpha.$$

348. Let i be the normal-vector to any plane.

Let ϖ and ρ be the vectors of any two points in a rigid plate in contact with the plane.

After any small displacement of the rigid plate in its plane, let $d\varpi$ and $d\rho$ be the increments of ϖ and ρ.

Then $Sid\varpi = 0$, $Sid\rho = 0$; and, since $T(\varpi-\rho)$ is constant,

$$S(\varpi-\rho)(d\varpi-d\rho) = 0.$$

And we may evidently assume

$$d\rho = \omega i(\rho-\tau),$$

$$d\varpi = \omega i(\varpi-\tau);$$

where of course τ is the vector of some point in the plane, to a rotation ω about which the displacement is therefore equivalent.

Eliminating it, we have

$$\omega i = \frac{d(\varpi-\rho)}{\varpi-\rho},$$

which gives ω, and thence τ is at once found.

For any other point σ in the plane figure

$$Sid\sigma = 0,$$

$$S(\rho-\sigma)(d\rho-d\sigma) = 0. \quad \text{Hence } d\rho-d\sigma = \omega_1 i(\rho-\sigma).$$

$$S(\sigma-\varpi)(d\varpi-d\sigma) = 0. \quad \text{Hence } d\sigma-d\varpi = \omega_2 i(\sigma-\varpi).$$

From which, at once, $\omega_1 = \omega_2 = \omega$, and

$$d\sigma = \omega i(\sigma-\tau),$$

or this point also is displaced by a rotation ω about an axis through the extremity of τ and parallel to i.

349. In the case of a rigid body moving about a fixed point let ϖ, ρ, σ denote the vectors of any three points of the body; the fixed point being origin.

Then ϖ^2, ρ^2, σ^2 are constant, and so are $S\varpi\rho$, $S\rho\sigma$, and $S\sigma\varpi$.

After any small displacement we have, for ϖ and ρ,

$$\left.\begin{aligned} S\varpi d\varpi &= 0, \\ S\rho d\rho &= 0, \\ S\varpi d\rho + S\rho d\varpi &= 0. \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now these three equations are satisfied by

$$d\varpi = V\alpha\varpi, \quad d\rho = V\alpha\rho,$$

where α is *any* vector whatever. But if $d\varpi$ and $d\rho$ are *given*, then

$$Vd\varpi d\rho = V.V\alpha\varpi V\alpha\rho = \alpha S.\alpha\rho\varpi.$$

Operate by $S.V\varpi\rho$, and remember (1),

$$S^2\varpi d\rho = S^2\rho d\varpi = S^2.\alpha\rho\varpi.$$

Hence
$$\alpha = \frac{Vd\varpi d\rho}{S\varpi d\rho} = \frac{Vd\rho d\varpi}{S\rho d\varpi} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Now consider σ,
$$\left.\begin{aligned} S\sigma d\sigma &= 0, \\ S\rho d\sigma &= -S\sigma d\rho, \\ S\varpi d\sigma &= -S\sigma d\varpi. \end{aligned}\right\}$$

$d\sigma = V\alpha\sigma$ satisfies them all, by (2), and we have thus the proposition that *any small displacement of a rigid body about a fixed point is equivalent to a rotation.*

350. *To represent the rotation of a rigid body about a given axis, through a given finite angle.*

Let α be a unit-vector in the direction of the axis, ρ the vector of any point in the body with reference to a fixed point in the axis, and θ the angle of rotation.

Then
$$\rho = \alpha^{-1}S\alpha\rho + \alpha^{-1}V\alpha\rho,$$
$$= -\alpha S\alpha\rho - \alpha V\alpha\rho.$$

The rotation leaves, of course, the first part unaffected, but the second evidently becomes
$$-\alpha^{\frac{2\theta}{\pi}}\alpha V\alpha\rho,$$
or
$$-\alpha V\alpha\rho\cos\theta + V\alpha\rho\sin\theta.$$

Hence ρ becomes
$$\rho_1 = -\alpha S\alpha\rho - \alpha V\alpha\rho\cos\theta + V\alpha\rho\sin\theta,$$
$$= \left(\cos\frac{\theta}{2} + \alpha\sin\frac{\theta}{2}\right)\rho\left(\cos\frac{\theta}{2} - \alpha\sin\frac{\theta}{2}\right),$$
$$= \alpha^{\frac{\theta}{\pi}}\rho\alpha^{-\frac{\theta}{\pi}}.$$

351. Hence *to compound two rotations about axes which meet*, we may evidently write, as the effect of an additional rotation ϕ about the unit-vector β,
$$\rho_2 = \beta^{\frac{\phi}{\pi}}\rho_1\beta^{-\frac{\phi}{\pi}}.$$

Hence
$$\rho_2 = \beta^{\frac{\phi}{\pi}}\alpha^{\frac{\theta}{\pi}}\rho\alpha^{-\frac{\theta}{\pi}}\beta^{-\frac{\phi}{\pi}}.$$

If the β-rotation had been first, and then the α-rotation, we should have had
$$\rho'_2 = \alpha^{\frac{\theta}{\pi}}\beta^{\frac{\phi}{\pi}}\rho\beta^{-\frac{\phi}{\pi}}\alpha^{-\frac{\theta}{\pi}},$$
and the non-commutative property of quaternion multiplication shows that we have *not*, in general,
$$\rho'_2 = \rho_2.$$

If α, β, γ be radii of the unit sphere to the corners of a spherical triangle whose angles are $\frac{\theta}{2}$, $\frac{\phi}{2}$, $\frac{\psi}{2}$, we know that
$$\gamma^{\frac{\psi}{\pi}}\beta^{\frac{\phi}{\pi}}\alpha^{\frac{\theta}{\pi}} = -1. \quad \text{(Hamilton, \textit{Lectures}, p. 267.)}$$

Hence $$\beta^{\frac{\phi}{\pi}}\alpha^{\frac{\theta}{\pi}} = -\gamma^{-\frac{\psi}{\pi}},$$

and we may write $$\rho_2 = \gamma^{-\frac{\psi}{\pi}}\rho\gamma^{\frac{\psi}{\pi}},$$

or, *successive rotations about radii to two corners of a spherical triangle, and through angles double of those of the triangle, are equivalent to a single rotation about the radius to the third corner, and through an angle double of the exterior angle of the triangle.*

Thus any number of successive *finite* rotations may be compounded into a single rotation about a definite axis.

352. When the rotations are indefinitely small, the effect of one is, by § 350, $$\rho_1 = \rho + \theta V\alpha\rho,$$

and for the two, neglecting products of small quantities,

$$\rho_2 = \rho + \theta V\alpha\rho + \phi V\beta\rho,$$

θ and ϕ representing the angles of rotation about the unit vectors α and β respectively.

But this is equivalent to

$$\rho_2 = \rho + T(\theta\alpha + \phi\beta)\,VU(\theta\alpha + \phi\beta)\,\rho,$$

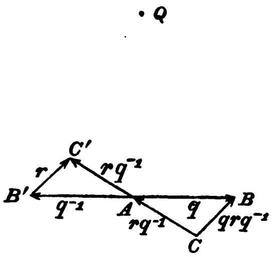
representing a rotation through an angle $T(\theta\alpha + \phi\beta)$, about the unit-vector $U(\theta\alpha + \phi\beta)$. Now the latter is the *direction*, and the former the *length*, of the diagonal of the parallelogram whose sides are $\theta\alpha$ and $\phi\beta$.

We may write these results more simply, by putting α for $\theta\alpha$, β for $\phi\beta$, where α and β are now no longer unit-vectors, but represent by their versors the *axes*, and by their tensors the *angles* (small), of rotation.

Thus $$\begin{aligned}\rho_1 &= \rho + V\alpha\rho,\\ \rho_2 &= \rho + V\alpha\rho + V\beta\rho,\\ &= \rho + V(\alpha + \beta)\,\rho.\end{aligned}$$

353. The general theorem, of which a few preceding sections illustrate special cases, is this:

By a rotation, about the axis of q, through double the angle of q, the quaternion r becomes the quaternion qrq^{-1}. Its tensor and angle remain unchanged, its plane or axis alone varies.



A glance at the figure is sufficient for the proof, if we note that of course $T.qrq^{-1} = Tr$, and therefore that we need consider the *versor* parts only. Let Q be the pole of q,

$\overset{\frown}{AB}=q$, $\overset{\frown}{AB'}=q^{-1}$, $\overset{\frown}{B'C'}=r$.

Join $C'A$, and make $\overset{\frown}{AC} = \overset{\frown}{C'A}$. Join CB.

Then $\overset{\frown}{CB}$ is qrq^{-1}, its arc CB is evidently equal in length to that of r, $B'C'$; and its plane (making the same angle with $B'B$ that that of $B'C'$ does) has evidently been made to revolve about Q, the pole of q, through double the angle of q.

If r be a vector, $= \rho$, then $q\rho q^{-1}$ is the result of a rotation through double the angle of q about the axis of q. Hence, as Hamilton has expressed it, if B represent a rigid system, or assemblage of vectors,

$$qBq^{-1}$$

is its new position after rotating through double the angle of q about the axis of q.

354. To compound such rotations, we have

$$r.qBq^{-1}.r^{-1} = rq.B.(rq)^{-1}.$$

To cause rotation through an angle t-fold the double of the angle of q we write

$$q^tBq^{-t}.$$

To reverse the direction of this rotation write

$$q^{-t}Bq^t.$$

To *translate* the body B without rotation, each point of it moving through the vector α, we write

$$\alpha+B.$$

To produce rotation of the translated body about the same axis, and through the same angle, as before,

$$q(\alpha+B)q^{-1}.$$

Had we rotated first, and then translated, we should have had

$$a+qBq^{-1}.$$

The discrepance between these last results might perhaps be useful to those who do not believe in the Moon's rotation, but to such men quaternions are unintelligible.

355. By the definition of *Homogeneous Strain*, it is evident that if we take any three (non-coplanar) unit-vectors a, β, γ in an unstrained mass, they become after the strain other vectors, not necessarily unit-vectors, a_1, β_1, γ_1.

Hence any other *given* vector, which of course may be thus expressed,

$$\rho = xa+y\beta+z\gamma,$$

becomes

$$\rho_1 = xa_1+y\beta_1+z\gamma_1,$$

and is therefore known if a_1, β_1, γ_1 be given.

More precisely

$$\rho S.a\beta\gamma = aS.\beta\gamma\rho+\beta S.\gamma a\rho+\gamma S.a\beta\rho$$

becomes

$$\rho_1 S.a\beta\gamma = \phi\rho S.a\beta\gamma = a_1 S.\beta\gamma\rho+\beta_1 S.\gamma a\rho+\gamma_1 S.a\beta\rho.$$

Thus the properties of ϕ, as in Chapter V, enable us to study with great simplicity strains or displacements in a solid or liquid.

For instance, *to find a vector whose direction is unchanged by the strain*, is to solve the equation

$$V\rho\phi\rho = 0, \quad \text{or} \quad \phi\rho = g\rho,$$

where g is a scalar unknown.

[This vector equation is equivalent to *three* simple equations, and contains only *three* unknown quantities; viz. *two* for the *direction* of ρ (the *tensor* does not enter, or, rather, is a factor of each side), and the unknown g.]

We have seen that every such equation leads to a cubic in g which may be written

$$g^3-m_2g^2+m_1g-m = 0,$$

where m_2, m_1, m are scalars depending in a known manner on the constant vectors involved in ϕ. This must have *one* real root, and may have *three*.

356. For simplicity let us assume that α, β, γ form a rectangular system, then we may operate by $S.\alpha$, $S.\beta$, and $S.\gamma$; and thus at once obtain the equation for g, in the form

$$\begin{vmatrix} S\alpha\alpha_1+g, & S\alpha\beta_1, & S\alpha\gamma_1 \\ S\beta\alpha_1, & S\beta\beta_1+g, & S\beta\gamma_1 \\ S\gamma\alpha_1, & S\gamma\beta_1, & S\gamma\gamma_1+g \end{vmatrix} = 0. \quad\ldots\ldots\ldots (1)$$

If the mass be rigid we must have α_1, β_1, γ_1 still rectangular unit-vectors, and therefore

$$\left.\begin{aligned} S\alpha\beta_1 &= S\beta\alpha_1, \\ S\alpha\gamma_1 &= S\gamma\alpha_1, \\ S\gamma\beta_1 &= S\beta\gamma_1, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which case (1) has obviously $+1$ as one root: the others being imaginary, except in two limiting cases in which their values are equal, and each is $+1$ or -1.

[One simple method of obtaining the conditions (2) is to suppose q the quaternion by a rotation about whose axis and through double of its angle α, β, γ are converted into α_1, β_1, γ_1. We have thus, by § 353,

$$\alpha_1 = q\alpha q^{-1} \text{ \&c.},$$

and equations (2) express the identities

$$S.\alpha q\beta q^{-1} = S.\beta q\alpha q^{-1} \text{ \&c.}$$

Numerous equally simple quaternion proofs will be obvious to the intelligent student.]

357. If we take $T\rho = C$ we consider a portion of the mass initially spherical. This becomes of course

$$T\phi^{-1}\rho_1 = C,$$

an ellipsoid, in the strained state of the body.

Or if we consider a portion which is spherical after the strain,

i. e. $$T\rho_1 = C,$$
its initial form was $$T\phi\rho = C,$$
another ellipsoid. The relation between these ellipsoids is obvious from their equations. (See 311.)

In either case the axes of the ellipsoid correspond to a rectangular set of three diameters of the sphere (§ 257). But we must carefully separate the cases in which these corresponding lines in the two surfaces are, and are not, coincident. For, in the former case there is *pure* strain, in the latter the strain is accompanied by rotation. Here we have at once the distinction pointed out by Stokes[*] and Helmholtz[†] between the cases of fluid motion in which there is, or is not, a velocity-potential. In ordinary fluid motion the distortion is of the nature of a pure strain, i. e. is differentially non-rotational; while in vortex motion it is essentially accompanied by rotation. But the resultant of two pure strains is generally a strain accompanied by rotation. The question before us beautifully illustrates the properties of the quaternion linear and vector function.

358. *To find the quaternion formula for a pure strain.* Take α, β, γ now as unit-vectors parallel to the axes of the strain-ellipsoid, they become $a\alpha, b\beta, c\gamma$.

Hence $$\rho_1 = \phi\rho = -a\alpha S\alpha\rho - b\beta S\beta\rho - c\gamma S\gamma\rho.$$

And we have, for the criterion of a pure strain, the property of the function ϕ, that it is *self-conjugate*, i. e.

$$S\rho\phi\sigma = S\sigma\phi\rho.$$

359. *Two pure strains, in succession, generally give a strain accompanied by rotation.* For if ϕ, ψ represent the strains, since they are pure we have

$$\left.\begin{aligned} S\rho\phi\sigma &= S\sigma\phi\rho, \\ S\rho\psi\sigma &= S\sigma\psi\rho. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots \quad (1)$$

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[*] *Cambridge Phil. Trans.* 1845.

[†] *Crelle,* vol. lv, 1857. See also *Phil. Mag.* (Supplement) June 1867.

But for the compound strain we have

$$\rho_1 = \chi\rho = \psi\phi\rho,$$

and we have *not* generally

$$S\rho\chi\sigma = S\sigma\chi\rho.$$

For $$S\rho\psi\phi\sigma = S\sigma\phi\psi\rho,$$

by (1), and $\psi\phi$ is not generally the same as $\phi\psi$. (See Ex. 7 to Chapter V.)

360. The simplicity of this view of the question might lead us to suppose that we may easily *separate the pure strain from the rotation in any case*, and exhibit the corresponding functions. But, for this purpose, it is generally necessary to solve the cubic equation of Chapter V in each particular case. When this is effected, the rest of the process presents no difficulty.

361. In general, if

$$\rho_1 = \phi\rho = -\alpha_1 S\alpha\rho - \beta_1 S\beta\rho - \gamma_1 S\gamma\rho,$$

the angle between any two lines, say ρ and σ, becomes in the altered state of the body

$$\cos^{-1}(-S.U\phi\rho U\phi\sigma).$$

The plane $S\zeta\rho = 0$ becomes (with the notation of § 144)

$$S\zeta\rho_1 = 0 = S\zeta\phi\rho = S\rho\phi'\zeta.$$

Hence the angle between the planes $S\zeta\rho = 0$, and $S\eta\rho = 0$, which is $\cos^{-1}(-S.U\zeta U\eta)$, becomes

$$\cos^{-1}(-S.U\phi'\zeta U\phi'\eta).$$

The *locus of lines equally elongated* is, of course,

$$T\phi U\rho = e,$$

or $$T\phi\rho = eT\rho,$$

a cone of the second order.

362. In the case of a *Simple Shear*, we have, obviously,

$$\rho_1 = \phi\rho = \rho + \beta S\alpha\rho.$$

The vectors which are unaltered in length are given by

$$T\rho_1 = T\rho,$$

or $$2S\beta\rho S\alpha\rho + \beta^2 S^2\alpha\rho = 0,$$

which breaks up into

$$S.\alpha\rho = 0,$$

and $$S\rho(2\beta + \beta^2\alpha) = 0.$$

The intersection of this plane with the plane of α, β is perpendicular to $2\beta + \beta^2\alpha$. Let it be $\alpha + x\beta$, then

$$S.(2\beta + \beta^2\alpha)(\alpha + x\beta) = 0,$$

i. e. $$2x - 1 = 0.$$

Hence the intersection required is

$$\alpha + \frac{\beta}{2}.$$

For the axes of the strain, one is of course $\alpha\beta$, and the others are found by making $T\phi U\rho$ a maximum and minimum.

Let $$\rho = \alpha + x\beta,$$

then $$\rho_1 = \phi\rho = \alpha + x\beta - \beta,$$

and $$\frac{T\rho_1}{T\rho} = \text{max. or min.},$$

gives $$x^2 - x + \frac{1}{\beta^2} = 0,$$

from which the values of x are found.

Also, as a verification,

$$S.(\alpha + x_1\beta)(\alpha + x_2\beta) = -1 + \beta^2 x_1 x_2,$$

and should be $= 0$. It is so, since, by the equation,

$$x_1 x_2 = \frac{1}{\beta^2}.$$

Again

$$S\{\alpha + (x_1 - 1)\beta\}\{\alpha + (x_2 - 1)\beta\} = -1 + \beta^2\{x_1 x_2 - (x_1 + x_2) + 1\},$$

which ought also to be zero. And, in fact, $x_1 + x_2 = 1$ by the equation; so that this also is verified.

363. We regret that our limits do not allow us to enter farther upon this very beautiful application.

But it may be interesting here, especially for the consideration of *any* continuous displacements of the particles of a mass, to introduce another of the extraordinary instruments of analysis which Hamilton has invented. Part of what is now to be given has been anticipated in last Chapter, but for continuity we commence afresh.

If
$$F\rho = C \text{ .... } (1)$$
be the equation of one of a system of surfaces, and if the differential of (1) be
$$S\nu d\rho = 0, \text{ .... } (2)$$
ν is a vector perpendicular to the surface, and *its length is inversely proportional to the normal distance between two consecutive surfaces.* In fact (2) shows that ν is perpendicular to $d\rho$, which is any tangent vector, thus proving the first assertion. Also, since in passing to a proximate surface we may write
$$S\nu\delta\rho = \delta C,$$
we see that
$$F(\rho + \nu^{-1}\delta C) = C + \delta C.$$
This proves the latter assertion.

It is evident from the above that if (1) be an equipotential, or an isothermal, surface, $-\nu$ *represents in direction and magnitude the force at any point or the flux of heat.* And we have seen (§ 317) that if
$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$
giving
$$\nabla^2 = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} - \frac{d^2}{dz^2},$$
then
$$\nu = \nabla F\rho.$$
This is due to Hamilton (*Lectures on Quaternions*, p. 611).

364. From this it follows that the effect of the vector operation ∇, upon any scalar function of the vector of a point, is to produce *the vector which represents in magnitude and direction the most rapid change in the value of the function.*

Let us next consider the effect of ∇ upon a *vector* function as

$$\sigma = i\xi + j\eta + k\zeta.$$

We have at once

$$\nabla\sigma = -\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) - i\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right) - \&c.$$

and in this semi-Cartesian form it is easy to see that :—

If σ represent a small vector displacement of a point situated at the extremity of the vector ρ (drawn from the origin)

$S\nabla\sigma$ represents the consequent cubical compression of the group of points in the vicinity of that considered, and

$V\nabla\sigma$ represents twice the vector axis of rotation of the same group of points.

Similarly

$$S\sigma\nabla = -\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right) = -D_\sigma,$$

or is equivalent to total differentiation in virtue of our having passed from one end to the other of the vector σ.

365. Suppose we fix our attention upon a group of points which originally filled a small sphere about the extremity of ρ as centre, whose equation referred to that point is

$$T\omega = e. \qquad (1)$$

After displacement ρ becomes $\rho + \sigma$, and, by last section, $\rho + \omega$ becomes $\rho + \omega + \sigma - (S\omega\nabla)\sigma$. Hence the vector of the new surface which encloses the group of points (drawn from the extremity of $\rho + \sigma$) is

$$\omega_1 = \omega - (S\omega\nabla)\sigma. \qquad (2)$$

Hence ω is a homogeneous linear and vector function of ω_1; or

$$\omega = \phi\omega_1,$$

and therefore, by (1),

$$T\phi\omega_1 = e,$$

the equation of the new surface, which is evidently a central surface of the second order, and therefore, of course, an ellipsoid.

We may solve (2) with great ease by approximation, if we remember that $T\sigma$ is very small, and therefore that in the small term we may put ω_1 for ω; i. e. omit squares of small quantities; thus,

$$\omega = \omega_1 + (S\omega_1\nabla)\sigma.$$

366. *If the small displacement of each point of a medium is in the direction of, and proportional to, the attraction exerted at that point by any system of material masses, the displacement is effected without rotation.*

For if $F\rho = C$ be the potential surface, we have $S\sigma d\rho$ a complete differential; i. e. in Cartesian cöordinates

$$\xi dx + \eta dy + \zeta dz$$

is a differential of three independent variables. Hence the vector axis of rotation

$$i\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) + \&c.,$$

vanishes by the vanishing of each of its constituents, or

$$V.\nabla\sigma = 0.$$

Conversely, *if there be no rotation, the displacements are in the direction of, and proportional to, the normal vectors to a series of surfaces.*

For $$0 = V.d\rho V.\nabla\sigma = (Sd\rho\nabla)\sigma - \nabla S\sigma d\rho,$$

where, in the last term, ∇ acts on σ alone.

Now, of the two terms on the right, the first is a complete differential, since it may be written $-D_{d\rho}\sigma$, and therefore the remaining term must be so.

Thus, in a distorted system, there is no compression if

$$S\nabla\sigma = 0,$$

and no rotation if

$$V.\nabla\sigma = 0;$$

and evidently merely transference if $\sigma = \alpha =$ a constant vector, which is one case of

$$\nabla\sigma = 0.$$

In the important case of

$$\sigma = e\nabla F\rho$$

there is evidently no rotation, since

$$\nabla\sigma = e\nabla^2 F\rho$$

is evidently a scalar. In this case, then, there are only translation and compression, and the latter is at each point proportional to the density of a distribution of matter, which would give the potential $F\rho$. For if r be such density, we have at once

$$\nabla^2 F\rho = 4\pi r$$[*].

367. The moment of inertia of a body about a unit vector α as axis is evidently

$$Mk^2 = -\Sigma m(V\alpha\rho)^2,$$

where ρ is the vector of the portion m of the mass, and the origin of ρ is in the axis.

Hence if we take $kT\alpha = e^2$, we have, as locus of the extremity of α,

$$Me^4 = -\Sigma m(V\alpha\rho)^2 = MS\alpha\phi\alpha \text{ (suppose)},$$

the momental ellipsoid.

If ϖ be the vector of the centre of inertia, σ the vector of m with respect to it, we have

$$\rho = \varpi + \sigma;$$

therefore

$$\begin{aligned} Mk^2 &= -\Sigma m\{(V\alpha\varpi)^2 + (V\alpha\sigma)^2\} \\ &= -M(V\alpha\varpi)^2 + MS\alpha\phi_1\alpha. \end{aligned}$$

Now, for principal axes, k is max., min., or max.-min, with the condition

$$\alpha^2 = -1.$$

Thus we have

$$\begin{aligned} &S\alpha'(\varpi V\alpha\varpi - \phi_1\alpha) = 0, \\ &S\alpha'\alpha = 0; \end{aligned}$$

therefore $-\phi_1\alpha + \varpi V\alpha\varpi = p\alpha = k^2\alpha$ (by operating by $S\alpha$).

Hence

$$(\phi_1 + k^2 + \varpi^2)\alpha = +\varpi S\alpha\varpi, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

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[*] *Proc. R. S. E.* 1862-3.

determines the values of α, k^2 being found from the equation

$$S\varpi(\phi+k^2+\varpi^2)^{-1}\varpi = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now the normal to

$$S\sigma(\phi+k^2+\varpi^2)^{-1}\sigma = 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

at the point σ is $\qquad (\phi+k^2+\varpi^2)^{-1}\sigma.$

But (3) passes through $-\varpi$, by (2), and *there* the normal is

$$(\phi+k^2+\varpi^2)^{-1}\varpi,$$

which, by (1), is parallel to one of the required values of α. Thus we prove Thomson's theorem that *the principal axes at any point are normals to the three surfaces, confocal with the momental ellipsoid, which pass through that point.*

EXAMPLES TO CHAPTER X.

1. Form, from kinematical principles, the equation of the cycloid; and employ it to prove the well-known elementary properties of the arc, tangent, radius of curvature, and evolute, of the curve.

2. Interpret, kinematically, the equation

$$\dot{\rho} = aU(\beta t-\rho),$$

where β is a given vector, and a a given scalar.

Show that it represents a plane curve; and give it in an integrated form independent of t.

3. If we write

$$\varpi = \beta t-\rho,$$

the equation in (2) becomes

$$\beta-\dot{\varpi} = aU\varpi.$$

Interpret this kinematically; and find an integral.

What is the nature of the step we have taken in transforming from the equation of (2) to that of the present question?

4. The motion of a point in a plane being given, refer it to

(*a.*) Fixed rectangular vectors in the plane.

(*b.*) Rectangular vectors in the plane, revolving uniformly about a fixed point.

(*c.*) Vectors, in the plane, revolving with different, but uniform, angular velocities.

(*d.*) The vector radius of a fixed circle, drawn to the point of contact of a tangent from the moving point.

In each case translate the result into Cartesian cöordinates.

5. Any point of a line of given length, whose extremities move in fixed lines in a given plane, describes an ellipse.

Show how to find the centre, and axes, of this ellipse; and the angular velocity about the centre of the ellipse of the tracing point when the describing line rotates uniformly.

Transform this construction so as to show that the ellipse is a hypotrochoid.

6. A point, A, moves uniformly round one circular section of a cone; find the angular velocity of the point, a, in which the generating line passing through A meets a subcontrary section, about the centre of that section.

7. Solve, generally, the problem of finding the path by which a point will pass in the least time from one given point to another, the velocity at the point of space whose vector is ρ being expressed by the given scalar function

$$f\rho.$$

Take also the following particular cases:—

(*a.*) $f\rho = a$ while $S\alpha\rho > 1$,
$f\rho = b$ while $S\alpha\rho < 1$.

(*b.*) $f\rho = S\alpha\rho$.

(*c.*) $f\rho = -\rho^2$.

8. If, in the preceding question, fp be such a function of $T\rho$ that any one swiftest path is a circle, every other such path is a circle, and all paths diverging from one point converge accurately in another.

(Maxwell, *Cam. and Dub. Math. Journal*, IX. p. 9.)

9. Interpret, as results of the composition of successive conical rotations, the apparent truisms

$$\frac{\alpha}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha} = 1,$$

and
$$\frac{\alpha}{\kappa}\frac{\kappa}{\iota}\frac{\iota}{\theta}\cdots\cdots\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha} = 1.$$

(Hamilton, *Lectures*, p. 334.)

10. Interpret, in the same way, the quaternion operators

$$q = (\delta\epsilon^{-1})^{\frac{1}{2}}(\epsilon\zeta^{-1})^{\frac{1}{2}}(\zeta\delta^{-1})^{\frac{1}{2}},$$

and
$$q = \left(\frac{\alpha}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{\delta}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}. \quad (\textit{Ibid.})$$

11. Find the axis and angle of rotation by which one given rectangular set of unit vectors α, β, γ is changed into another given set α_1, β_1, γ_1.

12. Show that, if

$$\phi\rho = \rho + V\epsilon\rho,$$

the linear and vector operation ϕ denotes rotation about the vector ϵ, together with uniform expansion in all directions perpendicular to it.

Prove this also by forming the operator which produces the expansion without the rotation, and that producing the rotation without the expansion; and finding their joint effect.

13. Express by quaternions the motion of a side of one right cone rolling uniformly upon another which is fixed, the vertices of the two being coincident.

14. Given the simultaneous angular velocities of a body about

the principal axes through its centre of inertia, find the position of these axes in space at any assigned instant.

15. Find the linear and vector function, and also the quaternion operator, by which we may pass, in any simple crystal of the cubical system, from the normal to one given face to that to another. How can we use them to distinguish a series of faces belonging to the same zone?

16. Classify the simple forms of the cubical system by the properties of the linear and vector function, or of the quaternion operator.

17. Find the vector normal of a face which truncates symmetrically the edge formed by the intersection of two given faces.

18. Find the normals of a pair of faces symmetrically truncating the given edge.

19. Find the normal of a face which is equally inclined to three given faces.

20. Show that the rhombic dodecahedron may be derived from the cube, or from the octahedron, by truncation of the edges.

21. Find the form whose faces replace, symmetrically, the edges of the rhombic dodecahedron.

22. Show how the two kinds of hemihedral forms are indicated by the quaternion expressions.

23. Show that the cube may be produced by truncating the edges of the regular tetrahedron.

24. Point out the modifications in the auxiliary vector function required in passing to the pyramidal and prismatic systems respectively.

25. In the rhombohedral system the auxiliary quaternion

operator assumes a singularly simple form. Give this form, and point out the results indicated by it.

26. Show that if the hodograph be a circle, and the acceleration be directed to a fixed point; the orbit must be a conic section, which is limited to being a circle if the acceleration follow any other law than that of gravity.

27. In the hodograph corresponding to acceleration $f(D)$ directed towards a fixed centre, the curvature is inversely as $D^2 f(D)$.

28. If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut by any two common orthogonals, the sum of the two times of hodographically describing the two intercepted arcs (small or large) will be the same for the two hodographs. (Hamilton, *Elements*, p. 725.)

29. Employ the last theorem to prove, after Lambert, that the time of describing any arc of an elliptic orbit may be expressed in terms of the chord of the arc and the extreme radii vectores.

CHAPTER XI.

PHYSICAL APPLICATIONS.

368. WE propose to conclude the work by giving a few instances of the ready applicability of quaternions to questions of mathematical physics, upon which, even more than on the Geometrical or Kinematical applications, the real usefulness of the Calculus must mainly depend—except, of course, in the eyes of that section of mathematicians for whom Transversals and Anharmonic Pencils, &c. have a to us incomprehensible charm. Of course we cannot attempt to give examples in all branches of physics, nor even to carry very far our investigations in any one branch: this Chapter is not intended to teach Physics, but merely to show by a few examples how expressly and naturally quaternions seem to be fitted for attacking the problems it presents.

We commence with a few general theorems in Dynamics—the formation of the equations of equilibrium and motion of a rigid system, some properties of the central axis, and the motion of a solid about its centre of inertia.

369. When any forces act on a rigid body, the force β at the point whose vector is α, &c., then, if the body be slightly displaced, so that α becomes $\alpha + \delta\alpha$, the whole work done is

$$\Sigma S\beta\delta\alpha.$$

This must vanish if the forces are such as to maintain equilibrium. Hence *the condition of equilibrium of a rigid body is*

$$\Sigma S\beta\delta\alpha = 0.$$

For a displacement of translation $\delta\alpha$ is *any* constant vector, hence

$$\Sigma\beta = 0. \quad \text{.............................} \quad (1)$$

For a rotation-displacement, we have by § 349, ϵ being the axis, and $T\epsilon$ being indefinitely small,

$$\delta\alpha = V\epsilon\alpha,$$

and $$\Sigma S.\beta V\epsilon\alpha = \Sigma S.\epsilon V\alpha\beta = S.\epsilon\Sigma(V\alpha\beta) = 0,$$

whatever be ϵ, hence

$$\Sigma . V\alpha\beta = 0. \quad \text{..........................} \quad (2)$$

These equations, (1) and (2), are equivalent to the ordinary six equations of equilibrium.

370. In general, for any set of forces, let

$$\Sigma\beta = \beta_1,$$
$$\Sigma . V\alpha\beta = \alpha_1,$$

it is required *to find the points for which the couple* α_1 *has its axis coincident with the resultant force* β_1. Let γ be the vector of such a point.

Then for it the axis of the couple is

$$\Sigma . V(\alpha-\gamma)\beta = \alpha_1 - V\gamma\beta_1,$$

and by condition

$$x\beta_1 = \alpha_1 - V\gamma\beta_1.$$

Operate by $S\beta_1$; therefore

$$x\beta_1^2 = S\alpha_1\beta_1,$$

and $$V\gamma\beta_1 = \alpha_1 - \beta_1^{-1}S\alpha_1\beta_1 = -\beta_1 V\alpha_1\beta_1^{-1},$$

or $$\gamma = V\alpha_1\beta_1^{-1} + y\beta_1,$$

a straight line (the *Central Axis*) parallel to the resultant force.

371. *To find the points about which the couple is least.*

Here $$T(\alpha_1 - V\gamma\beta_1) = \text{minimum}.$$

Therefore $$S.(\alpha_1 - V\gamma\beta_1)V\beta_1\gamma' = 0,$$

where γ' is any vector whatever. It is useless to try $\gamma' = \beta_1$, but we may put it in succession equal to α_1 and $V\alpha_1\beta_1$. Thus

$$S.\gamma V.\beta_1 V\alpha_1\beta_1 = 0,$$

and $$(V\alpha_1\beta_1)^2 - \beta_1^2 S.\gamma V\alpha_1\beta_1 = 0.$$

Hence $$\gamma = xV\alpha_1\beta_1 + y\beta_1,$$

and by operating with $S.V\alpha_1\beta_1$, we get

$$\frac{1}{\beta_1^2}(V\alpha_1\beta_1)^2 = x(V\alpha_1\beta_1)^2,$$

or $$\gamma = V\alpha_1\beta_1{}^{-1} + y\beta_1,$$

the same locus as in last section.

372. The couple vanishes if

$$\alpha_1 - V\gamma\beta_1 = 0.$$

This necessitates $$S\alpha_1\beta_1 = 0,$$

or the force must be *in* the plane of the couple. If this be the case,

$$\gamma = \alpha_1\beta_1{}^{-1} + x\beta_1,$$

still the central axis.

373. To assign the values of forces ξ, ξ_1, to act at ϵ, ϵ_1, and be equivalent to the given system.

$$\xi + \xi_1 = \beta_1,$$

$$V\epsilon\xi + V\epsilon_1\xi_1 = \alpha_1.$$

Hence $$V\epsilon\xi + V\epsilon_1(\beta_1 - \xi) = \alpha_1,$$

and $$\xi = (\epsilon - \epsilon_1)^{-1}(\alpha_1 - V\epsilon_1\beta_1) + x(\epsilon - \epsilon_1).$$

Similarly for ξ_1. The indefinite terms may be omitted, as they must evidently be equal and opposite. In fact they are any equal and opposite forces whatever acting in the line joining the given points.

374. For the motion of a rigid system, we have of course

$$\Sigma S(m\ddot{\alpha} - \beta)\delta\alpha = 0,$$

by the general equation of Lagrange.

Suppose the displacements $\delta\alpha$ to correspond to a mere *translation*, then $\delta\alpha$ is *any* constant vector, hence

$$\Sigma(m\ddot{\alpha}-\beta) = 0,$$

or, if α_1 be the vector of the centre of inertia, and therefore

$$\alpha_1\Sigma m = \Sigma m\alpha,$$

we have at once

$$\ddot{\alpha}_1\Sigma m - \Sigma\beta = 0,$$

and the centre of inertia moves as if the whole mass were concentrated in it, and acted upon by all the applied forces.

375. Again, let the displacements $\delta\alpha$ correspond to a rotation about an axis ϵ, passing through the origin, then

$$\delta\alpha = V\epsilon\alpha,$$

it being assumed that $T\epsilon$ is indefinitely small.

Hence $$\Sigma S.\epsilon V\alpha(m\ddot{\alpha}-\beta) = 0,$$

for *all* values of ϵ, and therefore

$$\Sigma.V\alpha(m\ddot{\alpha}-\beta) = 0,$$

which contains the three remaining ordinary equations of motion.

Transfer the origin to the centre of inertia, i. e. put $\alpha=\alpha_1+\varpi$, then our equation becomes

$$\Sigma V(\alpha_1+\varpi)(m\ddot{\alpha}_1+m\ddot{\varpi}-\beta) = 0.$$

Or, since $\Sigma m\varpi = 0$,

$$\Sigma V\varpi(m\ddot{\varpi}-\beta)+V\alpha_1(\ddot{\alpha}_1\Sigma m-\Sigma\beta) = 0.$$

But $\ddot{\alpha}_1\Sigma m-\Sigma\beta = 0$, hence our equation is simply

$$\Sigma V\varpi(m\ddot{\varpi}-\beta) = 0.$$

Now $\Sigma V\varpi\beta$ is the couple, about the centre of inertia, produced by the applied forces; call it ϕ, then

$$\Sigma mV\varpi\ddot{\varpi} = \phi. \quad \text{........................} \quad (1)$$

376. Integrating once,

$$\Sigma mV\varpi\dot{\varpi} = \gamma+\int\phi dt. \quad \text{.................} \quad (2)$$

Again, as the motion considered is *relative* to the centre of inertia, it must be of the nature of rotation about some axis, in general variable. Let ϵ denote at once the direction of, and the angular velocity about, this axis. Then, evidently,

$$\dot{\varpi} = V\epsilon\varpi.$$

Hence, the last equation may be written

$$\Sigma m\varpi V\epsilon\varpi = \gamma + \int\phi dt.$$

Operating by $S.\epsilon$, we get

$$\Sigma m(V\epsilon\varpi)^2 = S\epsilon\gamma + S\epsilon\int\phi dt. \qquad (3)$$

But, by operating directly by $2\int S\epsilon dt$ upon the equation (1), we get

$$\Sigma m(V\epsilon\varpi)^2 = -h^2 + 2\int S\epsilon\phi dt. \qquad (4)$$

(2) and (4) contain the usual four integrals of the first order.

377. When no forces act on the body, we have $\phi = 0$, and therefore

$$\Sigma m\varpi V\epsilon\varpi = \gamma, \qquad (5)$$

$$\Sigma m\dot{\varpi}^2 = \Sigma m(V\epsilon\varpi)^2 = -h^2, \qquad (6)$$

and, from (5) and (6),

$$S\epsilon\gamma = -h^2. \qquad (7)$$

One interpretation of (6) is, that the kinetic energy of rotation remains unchanged: another is, that the vector ϵ terminates in an ellipsoid whose centre is the origin, and which therefore assigns the angular velocity when the direction of the axis is given; (7) shows that the extremity of the instantaneous axis is always in a plane fixed in space.

Also, by (5), (7) is the equation of the tangent plane to (6) at the extremity of the vector ϵ. Hence the ellipsoid (6) *rolls* on the plane (7).

From (5) and (6), we have at once, as an equation which ϵ must satisfy,

$$\gamma^2\Sigma.m(V\epsilon\varpi)^2 = -h^2(\Sigma.m\varpi V\epsilon\varpi)^2.$$

This belongs to a cone of the second degree fixed in the body. Thus all the ordinary results regarding the motion of a rigid body under the action of no forces, the centre of inertia being fixed, are deduced almost intuitively: and the only difficulties to be met with in more complex cases of such motion are those of integration, which are inherent to the subject, and appear whatever analytical method is employed. (Hamilton, *Proc. R.I.A.* 1848.)

378. We next take Fresnel's Theory of Double Refraction, but merely for the purpose of showing how quaternions simplify the processes required, and in no way to discuss the plausibility of the physical assumptions.

Let $t\varpi$ be the vector displacement of a portion of the ether, with the condition

$$\varpi^2 = -1, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

the force of restitution, on Fresnel's assumption, is

$$t\,(a^2 iSi\varpi + b^2 jSj\varpi + c^2 kSk\varpi) = t\phi\varpi,$$

using the notation of Chapter V. Here the function ϕ is obviously self-conjugate. a^2, b^2, c^2 are optical constants depending on the crystalline medium, and on the colour of the light, and may be considered as given.

Fresnel's second assumption is that the ether is incompressible, or that vibrations normal to a wave front are inadmissible. If, then, α be the unit normal to a plane wave in the crystal, we have of course

$$\alpha^2 = -1, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\text{and} \qquad S\alpha\varpi = 0\,; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

but, and in addition, we have

$$\varpi^{-1}V\varpi\phi\varpi \parallel \alpha,$$

$$\text{or} \qquad S.\alpha\varpi\phi\varpi = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

This equation (4) is the embodiment of Fresnel's second assumption, but it may evidently be read as meaning, *the normal to the front, the direction of vibration, and that of the force of restitution are in one plane.*

379. Equations (3) and (4), if satisfied by ϖ, are also satisfied by $\varpi\alpha$, so that the plane (3) intersects the cone (4) in two lines at right angles to each other. That is, *for any given wave front there are two directions of vibration, and they are perpendicular to each other.*

380. The square of the normal velocity of propagation of a plane wave is proportional to the ratio of the resolved part of the force of restitution in the direction of vibration, to the amount of displacement, hence

$$v^2 = S\varpi\phi\varpi.$$

Hence Fresnel's *Wave-surface* is the envelop of the plane

$$S\alpha\rho = \sqrt{S\varpi\phi\varpi}, \quad \dots\dots \quad (5)$$

with the conditions

$$\varpi^2 = -1, \quad \dots\dots \quad (1)$$

$$\alpha^2 = -1, \quad \dots\dots \quad (2)$$

$$S\alpha\varpi = 0, \quad \dots\dots \quad (3)$$

$$S.\alpha\varpi\phi\varpi = 0. \quad \dots\dots \quad (4)$$

Formidable as this problem appears, it is easy enough. From (3) and (4) we get at once,

$$x\varpi = V.\alpha V\alpha\phi\varpi,$$

Hence, operating by $S.\varpi$,

$$-x = -S\varpi\phi\varpi = -v^2.$$

Therefore

$$(\phi+v^2)\varpi = -\alpha S\alpha\phi\varpi,$$

and

$$S.\alpha(\phi+v^2)^{-1}\alpha = 0. \quad \dots\dots \quad (6)$$

In passing, we may remark that *this equation gives the normal velocities of the two rays whose fronts are perpendicular to* α. In Cartesian cöordinates it is the well-known equation

$$\frac{l^2}{a^2-v^2} + \frac{m^2}{b^2-v^2} + \frac{n^2}{c^2-v^2} = 0.$$

By this elimination of ϖ, our equations are reduced to

$$S.\alpha(\phi+v^2)^{-1}\alpha = 0, \quad \dots\dots \quad (6)$$

$$v = -S\alpha\rho, \quad \dots\dots \quad (5)$$

$$\alpha^2 = -1. \quad \dots\dots \quad (2)$$

They give at once, by § 309,

$$(\phi+v^2)^{-1}\alpha+v\rho\, S\alpha(\phi+v^2)^{-2}\alpha = k\alpha.$$

Operating by $S.\alpha$ we have

$$v^2 S\alpha(\phi+v^2)^{-2}\alpha = k.$$

Substituting for k, and remarking that

$$S\alpha\,(\phi+v^2)^{-2}\alpha = -T^2(\phi+v^2)^{-1}\alpha,$$

because ϕ is self-conjugate, we have

$$v\,(\phi+v^2)^{-1}\alpha = \frac{v\alpha-\rho}{\rho^2+v^2}.$$

This gives at once, by rearrangement,

$$v(\phi+v^2)^{-1}\alpha = (\phi-\rho^2)^{-1}\rho.$$

Hence $(\phi-\rho^2)^{-1}\rho = \dfrac{v\alpha-\rho}{\rho^2+v^2}.$

Operating by $S.\rho$ on this equation we have

$$S\rho(\phi-\rho^2)^{-1}\rho = -1, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

which is the required equation.

[It will be a good exercise for the student to translate the last ten formulae into Cartesian coördinates. He will thus reproduce almost exactly the steps by which Archibald Smith * first arrived at a simple and symmetrical mode of effecting the elimination. Yet, as we shall presently see, the above process is far from being the shortest and easiest to which quaternions conduct us.]

381. The Cartesian form of the equation (7) is not the usual one. It is, of course,

$$\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2} = -1.$$

But write (7) in the form

$$S.\rho\frac{\rho^2}{\phi-\rho^2}\rho = -\rho^2,$$

or

$$S.\rho\frac{\phi}{\phi-\rho^2}\rho = 0,$$

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* *Cambridge Phil. Trans.* 1843.

and we have the usual expression

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0.$$

This last quaternion equation can also be put into either of the new forms

$$T\left(\frac{\phi}{\phi-\rho^2}\right)^{\frac{1}{2}}\rho=0,$$

or $$T(\rho^{-2}-\phi^{-1})^{-\frac{1}{2}}\rho=0.$$

382. By applying the results of §§ 171, 172 we may introduce a multitude of new forms. We must confine ourselves to the most simple; but the student may easily investigate others by a process precisely similar to that which follows.

Writing the equation of the wave as

$$S\rho(\phi^{-1}+g)^{-1}\rho=0,$$

where we have $$g=-\rho^{-2},$$

we see that it may be changed to

$$S\rho(\phi^{-1}+h)^{-1}\rho=0,$$

if $$mS\rho\phi\rho=gh\rho^2=-h.$$

Thus the new form is

$$S\rho(\phi^{-1}-mS\rho\phi\rho)^{-1}\rho=0. \qquad (1)$$

Here $$m=\frac{1}{a^2b^2c^2}, \qquad S\rho\phi\rho=a^2x^2+b^2y^2+c^2z^2,$$

and the equation of the wave in Cartesian cöordinates is, putting

$$r_1^2=a^2x^2+b^2y^2+c^2z^2,$$

$$\frac{x^2}{b^2c^2-r_1^2}+\frac{y^2}{c^2a^2-r_1^2}+\frac{z^2}{a^2b^2-r_1^2}=0.$$

383. By means of equation (1) of last section we may easily prove Plücker's Theorem. *The Wave-Surface is its own reciprocal with respect to the ellipsoid whose equation is*

$$S\rho\phi^{\frac{1}{2}}\rho=\frac{1}{\sqrt{m}}.$$

The equation of the plane of contact of tangents to this surface from the point whose vector is ρ is

$$S\varpi\phi^{\frac{1}{2}}\rho = \frac{1}{\sqrt{m}}\cdot$$

The reciprocal of this plane, with respect to the unit-sphere about the origin, has therefore a vector σ where

$$\sigma = \sqrt{m}\,\phi^{\frac{1}{2}}\rho.$$

Hence $$\rho = \frac{1}{\sqrt{m}}\phi^{-\frac{1}{2}}\sigma,$$

and when this is substituted in the equation of the wave we have for the reciprocal (with respect to the unit-sphere) of the reciprocal of the wave with respect to the above ellipsoid,

$$S.\sigma\left(\phi - \frac{1}{m}S\sigma\phi^{-1}\sigma\right)\sigma = 0.$$

This differs from the equation (1) of last section solely in having ϕ^{-1} instead of ϕ, and (consistently with this) $\frac{1}{m}$ instead of m. Hence it represents the index-surface. The required reciprocal of the wave with reference to the ellipsoid is therefore the wave itself.

384. Hamilton has given a remarkably simple investigation of the form of the equation of the wave-surface, in his *Elements* p. 736, which the reader may consult with advantage. The following is essentially the same, but several steps of the process, which a skilled analyst would not require to write down, are retained for the benefit of the learner.

Let $$S\mu\rho = -1 \qquad\qquad (1)$$

be the equation of any tangent plane to the wave, i.e. of any wave-front. Then μ is the vector of wave-slowness, and the normal velocity of propagation is therefore $\frac{1}{T\mu}$. Hence, if ϖ be the vector direction of displacement, $\mu^{-2}\varpi$ is the effective component of the force of restitution. Hence, $\phi\varpi$ denoting the whole force of restitution, we have

$$\phi\varpi - \mu^{-2}\varpi \parallel \mu,$$

$$\text{or} \qquad \varpi \parallel (\phi - \mu^{-2})^{-1}\mu,$$

and, as ϖ is in the plane of the wave-front,

$$S\mu\varpi = 0,$$

$$\text{or} \qquad S\mu(\phi - \mu^{-2})^{-1}\mu = 0. \quad\ldots\ldots\ldots\ldots \quad (2)$$

This is, in reality, equation (6) of § 380. It appears here, however, as the equation of the *Index-Surface*, the polar reciprocal of the wave with respect to a unit-sphere about the origin. Of course the optical part of the problem is now solved, all that remains being the geometrical process of § 311.

385. Equation (2) of last section may be at once transformed, by the process of § 381, into

$$S\mu(\mu^2 - \phi^{-1})^{-1}\mu = 1.$$

Let us employ an auxiliary vector

$$\tau = (\mu^2 - \phi^{-1})^{-1}\mu,$$

$$\text{whence} \qquad \mu = (\mu^2 - \phi^{-1})\tau. \quad\ldots\ldots\ldots\ldots \quad (1)$$

The equation now becomes

$$S\mu\tau = 1, \quad\ldots\ldots\ldots\ldots \quad (2)$$

$$\text{or, by (1),} \qquad \mu^2\tau^2 - S\tau\phi^{-1}\tau = 1. \quad\ldots\ldots\ldots\ldots \quad (3)$$

Differentiating (3), subtract its half from the result obtained by operating with $S.\tau$ on the differential of (1). The remainder is

$$\tau^2 S\mu d\mu - S\tau d\mu = 0.$$

But we have also (§ 311)

$$S\rho d\mu = 0,$$

$$\text{and therefore} \qquad x\rho = \mu\tau^2 - \tau,$$

where x is a scalar.

This equation, with (2), shows that

$$S\tau\rho = 0. \quad\ldots\ldots\ldots\ldots \quad (4)$$

Hence, operating on it by $S.\rho$, we have by (1) of last section

$$x\rho^2 = -\tau^2,$$

and therefore $$\rho^{-1} = -\mu+\tau^{-1}.$$

This gives $$\rho^{-2} = \mu^2-\tau^{-2}.$$

Substituting from these equations in (1) above, it becomes

$$\tau^{-1}-\rho^{-1} = (\rho^{-2}+\tau^{-2}-\phi^{-1})\tau,$$

or $$\tau = (\phi^{-1}-\rho^{-2})^{-1}\rho^{-1}.$$

Finally, we have for the required equation, by (4),

$$S\rho^{-1}(\phi^{-1}-\rho^{-2})^{-1}\rho^{-1} = 0,$$

or, by a transformation already employed,

$$S\rho(\phi-\rho^2)^{-1}\rho = -1.$$

386. It may assist the student in the *practice* of quaternion analysis, which is our main object, if we give a few of these investigations by a somewhat varied process.

Thus, in § 378, let us write as in § 168,

$$a^2 iSi\varpi+b^2 jSj\varpi+c^2 kSk\varpi = \lambda'S\mu'\varpi+\mu'S\lambda'\varpi-p'\varpi.$$

We have, by the same processes as in § 378,

$$S.\varpi a\lambda'S\mu'\varpi+S.\varpi a\mu'S\lambda'\varpi = 0.$$

This may be written, *so far as the generating lines we require are concerned*,

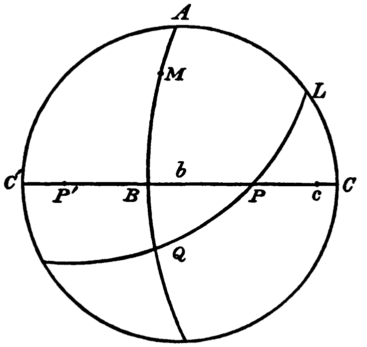
$$\left.\begin{array}{l} \qquad S.\varpi aV.\lambda'\varpi\mu' = 0 = S.\varpi a\lambda'\varpi\mu', \\ \text{since } \varpi a \text{ is a vector.} \\ \text{Or we may write} \\ \qquad S.\mu'V.\varpi\lambda'\varpi a = 0 = S.\mu'\varpi\lambda'\varpi a. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (1)$$

Equations (1) denote two cones of the second order which pass through the intersections of (3) and (4) of § 378. Hence their intersections are the directions of vibration.

387. By (1) we have

$$S.\varpi\lambda'\varpi a\mu' = 0.$$

Hence $\varpi\lambda'\varpi$, a, μ' are coplanar; and, as ϖ is perpendicular to a, it is equally inclined to $V\lambda' a$ and $V\mu' a$.



For, if L, M, A be the projections of λ', μ', a on the unit sphere, BC the great circle whose pole is A, we are to find for the projections of the values of ϖ on the sphere points P and P', such that if LP be produced till $\widehat{PQ} = \widehat{LP}$, Q may lie on the great circle AM. Hence, evidently,

$$\widehat{CP} = \widehat{PB},$$

and $\widehat{C'P'} = \widehat{P'B}$; which proves the proposition, since the projections of $V\lambda' a$ and $V\mu' a$ on the sphere are points b and c in BC, distant by quadrants from C and B respectively.

388. Or thus, $$S\varpi a = 0,$$

$$S.\varpi V.a\lambda'\varpi\mu' = 0,$$

therefore $$x\varpi = V.aV.a\lambda'\varpi\mu',$$

$$= -V.\lambda'\varpi\mu' - aSaV.\lambda'\varpi\mu'.$$

Hence $$(S\lambda'\mu' - x)\varpi = (\lambda' + aSa\lambda')S\mu'\varpi + (\mu' + aSa\mu')S\lambda'\varpi.$$

Operate by $S.\lambda'$, and we have

$$(x + S\lambda' aS\mu' a)S\lambda'\varpi = [\lambda'^2 a^2 - S^2\lambda' a]S\mu'\varpi$$

$$= S\mu'\varpi T^2 V\lambda' a.$$

Hence by symmetry,

$$\frac{S\mu'\varpi}{S\lambda'\varpi} T^2 V\lambda' a = \frac{S\lambda'\varpi}{S\mu'\varpi} T^2 V\mu' a,$$

or $$\frac{S\lambda'\varpi}{TV\lambda' a} \pm \frac{S\mu'\varpi}{TV\mu' a} = 0,$$

$$S\varpi\left(\frac{\lambda'}{TV\lambda' a} \pm \frac{\mu'}{TV\mu' a}\right) = 0,$$

and as $$S\varpi a = 0\ ,$$

$$\varpi = U(UV\lambda' a \pm UV\mu' a).$$

389. The optical interpretation of the common result of the last two sections is that *the planes of polarization of the two rays whose wave-fronts are parallel, bisect the angles contained by planes passing through the normal to the wave-front and the vectors* (optic axes) λ', μ'.

390. As in § 380, the normal velocity is given by

$$v^2 = S\varpi\phi\varpi = 2\,S\lambda'\varpi S\mu'\varpi - p'\varpi^2$$

$$= p' \mp \frac{S^2.\lambda'\mu'a}{(T\mp S).V\lambda'aV\mu'a}.$$

[This transformation, effected by means of the value of ϖ in § 388, is left to the reader.]

Hence, if v_1, v_2, be the velocities of the two waves whose normal is a

$$v_1^2 - v_2^2 = 2T.V\lambda'aV\mu'a$$

$$\propto \sin\widehat{\lambda'a}\sin\widehat{\mu'a}.$$

That is, *the difference of the squares of the velocities of the two waves varies as the product of the sines of the angles between the normal to the wave-front and the optic axes* (λ', μ').

391. We have, obviously,

$$(T^2 - S^2).V\lambda'a\,V\mu'a = T^2V.V\lambda'a\,V\mu'a = S.^2\lambda'\mu'a.$$

Hence $$v^2 = p' \mp (T \pm S).V\lambda'a\,V\mu'a.$$

The equation of the index surface, for which

$$T\rho = \frac{1}{v}, \qquad U\rho = a,$$

is therefore

$$1 = -p'\rho^2 \mp (T \pm S).V\lambda'\rho\,V\mu'\rho.$$

This will, of course, become the equation of the reciprocal of the index-surface, i. e. the wave-surface, if we put for the function ϕ its reciprocal: i. e. if in the values of λ', μ', p' we put $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ for a, b, c respectively. We have then, and indeed it might have been deduced even more simply as a transformation of § 380 (7),

$$1 = -p\rho^2 \mp (T \pm S).V\lambda\rho V\mu\rho,$$

as another form of the equation of Fresnel's wave.

If we employ the ι, κ transformation of § 121, this may be written, as the student may easily prove, in the form

$$(\kappa^2-\iota^2)^2 = S^2(\iota-\kappa)\rho + (TV\iota\rho \mp TV\kappa\rho)^2.$$

392. We may now, in furtherance of our object, which is to give varied examples of quaternions, not complete treatment of any one subject, proceed to deduce some of the properties of the wave-surface from the different forms of its equation which we have given.

393. *Fresnel's construction of the wave by points.*

From § 273 (4) we see at once that the lengths of the principal semidiameters of the central section of the ellipsoid

$$S\rho\phi^{-1}\rho = 1,$$

by the plane

$$S\alpha\rho = 0,$$

are determined by the equation

$$S.\alpha(\phi^{-1}-\rho^{-2})^{-1}\alpha = 0.$$

If these lengths be laid off along α, the central perpendicular to the cutting plane, their extremities lie on a surface for which $\alpha = U\rho$, and $T\rho$ has values determined by the equation.

Hence the equation of the locus is

$$S\rho(\phi^{-1}-\rho^{-2})^{-1}\rho = 0,$$

as in §§ 380, 385.

Of course the index-surface is derived from the reciprocal ellipsoid

$$S\rho\phi\rho = 1$$

by the same construction.

394. Again, in the equation

$$1 = -p\rho^2 \mp (T \pm S).V\lambda\rho V\mu\rho,$$

suppose $V\lambda\rho = 0,$ or $V\mu\rho = 0,$
we obviously have

$$\rho = \pm \frac{U\lambda}{\sqrt{p}} \quad \text{or} \quad \rho = \pm \frac{U\mu}{\sqrt{p}},$$

and there are therefore four singular points.

To find the nature of the surface near these points put

$$\rho = \frac{U\lambda}{\sqrt{p}} + \varpi,$$

where $T\varpi$ is very small, and reject terms above the first order in $T\varpi$. The equation of the wave becomes, in the neighbourhood of the singular point,

$$2pS\lambda\varpi - S.\varpi V.\lambda V\lambda\mu = \pm T.V\lambda\varpi V\lambda\mu,$$

which belongs to a cone of the second order.

395. From the similarity of its equation to that of the wave, it is obvious that the index-surface also has four conical cusps. As an infinite number of tangent planes can be drawn at such a point, the reciprocal surface must be capable of being touched by a plane at an infinite number of points; so that the wave-surface has four tangent planes which touch it along ridges.

To find their form, let us employ the last form of equation of the wave in § 391. If we put

$$TV\iota\rho = TV\kappa\rho, \text{..........................} \quad (1)$$

we have the equation of a cone of the second degree. It meets the wave at its intersections with the planes

$$S(\iota-\kappa)\rho = \pm (\kappa^2 - \iota^2). \quad \text{.....................} \quad (2)$$

Now the wave-surface is *touched* by these planes, because we cannot have the quantity on the first side of this equation greater in absolute magnitude than that on the second, so long as ρ satisfies the equation of the wave.

That the curves of contact are circles appears at once from (1) and (2), for they give in combination

$$\rho^2 = \mp S(\iota+\kappa)\rho, \text{ ....} \quad (3)$$

the equations of two spheres on which the curves in question are situated.

The diameter of this circular ridge is

$$TV.(\iota+\kappa)U(\iota-\kappa) = \frac{2TV\iota\kappa}{T(\iota-\kappa)} = \frac{1}{b}\sqrt{(a^2-b^2)(b^2-c^2)}.$$

[Simple as these processes are, the student will find on trial that the equation

$$S\rho(\phi^{-1}-\rho^{-2})^{-1}\rho = 0,$$

gives the results quite as simply. For we have only to examine the cases in which $-\rho^{-2}$ has the value of one of the roots of the symbolical cubic in ϕ^{-1}. In the present case $T\rho=b$ is the only one which requires to be studied.]

396. By § 384, we see that the auxiliary vector of the succeeding section, viz.

$$\tau = (\mu^2-\phi^{-1})^{-1}\mu = (\phi^{-1}-\rho^{-2})^{-1}\rho^{-1},$$

is parallel to the direction of the force of restitution, $\phi\varpi$. Hence, as Hamilton has shown, the equation of the wave, in the form

$$S\tau\rho = 0,$$

(4) of § 385, indicates that *the direction of the force of restitution is perpendicular to the ray.*

Again, as for any one versor of a vector of the wave there are two values of the tensor, which are found from the equation

$$S.U\rho(\phi^{-1}-\rho^{-2})^{-1}U\rho = 0,$$

we see by § 393 that *the lines of vibration for a given plane front are parallel to the axes of any section of the ellipsoid*

$$S.\rho\phi^{-1}\rho = 1$$

made by a plane parallel to the front; or to the tangents to the lines of curvature at a point where the tangent plane is parallel to the wave-front.

397. Again, *a curve which is drawn on the wave-surface so as to touch at each point the corresponding line of vibration* has

$$\phi d\rho \parallel (\phi^{-1}-\rho^{-2})^{-1}\rho.$$

Hence $S\phi\rho d\rho = 0$, or $S\rho\phi\rho = C$,

so that such curves are the intersections of the wave with a series of ellipsoids concentric with it.

398. For *curves cutting at right angles the lines of vibration* we have

$$d\rho \parallel V\rho\phi^{-1}(\phi^{-1}-\rho^{-2})^{-1}\rho$$
$$\parallel V\rho(\phi-\rho^{2})^{-1}\rho.$$

Hence $S\rho d\rho = 0$, or $T\rho = C$,

so that the curves in question lie on concentric spheres.

They are also *spherical conics*, because where

$$T\rho = C$$

the equation of the wave becomes

$$S.\rho(\phi^{-1}+C^{-2})^{-1}\rho = 0,$$

the equation of a cyclic cone, whose vertex is at the common centre of the sphere and the wave-surface, and which cuts them in their curve of intersection.

399. As a final example we take the case of the action of electric currents on one another or on magnets; and the mutual action of permanent magnets.

A comparison between the processes we employ and those of Ampère (*Théorie des Phénomènes Electrodynamiques, &c.*, many of which are well given by Murphy in his *Electricity*) will at once show how much is gained in simplicity and directness by the use of quaternions.

The same gain in simplicity will be noticed in the investigations of the mutual effects of permanent magnets, where the resultant forces and couples are at once introduced in their most natural and direct forms.

400. Ampère's experimental laws may be stated as follows:

I. Equal and opposite currents in the same conductor produce equal and opposite effects on other conductors: whence it follows that an element of one current has no effect on an element of another which lies in the plane bisecting the former at right angles.

II. The effect of a conductor bent or twisted in any manner is equivalent to that of a straight one, provided that the two are traversed by equal currents, and the former *nearly* coincides with the latter.

III. No closed circuit can set in motion an element of a circular conductor about an axis through the centre of the circle and perpendicular to its plane.

IV. In similar systems traversed by equal currents the forces are equal.

To these we add the assumption that the action between two elements of currents is in the straight line joining them: and two others, viz. that the effect of any element of a current on another is directly as the product of the strengths of the currents, and of the lengths of the elements.

401. Let there be two closed currents whose strengths are a and a_1; let α', α_1 be elements of these, α being the vector joining their middle points. Then the effect of α' on α_1 must, when resolved along α_1, be a complete differential with respect to α (i. e. with respect to the three independent variables involved in α), since the total resolved effect of the closed circuit of which α' is an element is zero by III.

Also by I,, II, the effect is a function of $T\alpha$, $S\alpha\alpha'$, $S\alpha\alpha_1$, and $S\alpha'\alpha_1$, since these are sufficient to resolve α' and α_1 into elements parallel and perpendicular to each other and to α. Hence the mutual effect is

$$aa_1 U\alpha f(T\alpha, S\alpha\alpha', S\alpha\alpha_1, S\alpha'\alpha_1),$$

and the resolved effect parallel to α_1 is

$$aa_1 SU\alpha_1 U\alpha f.$$

Also, that action and reaction may be equal in absolute magnitude, f must be symmetrical in $S\alpha\alpha'$ and $S\alpha\alpha_1$. Again, α' (as differential of α) *can* enter *only to the first power*, and *must* appear in each term of f.

Hence $$f = AS\alpha'\alpha_1 + BS\alpha\alpha' S\alpha\alpha_1.$$

But, by IV, this must be independent of the dimensions of the system. Hence A is of -2 and B of -4 dimensions in $T\alpha$. Therefore

$$\frac{1}{T\alpha}\{AS\alpha\alpha_1 S\alpha'\alpha_1 + BS\alpha\alpha' S^2\alpha\alpha_1\}$$

is a complete differential, with respect to α, if $d\alpha = \alpha'$. Let

$$A = \frac{C}{T\alpha^2},$$

where C is a constant depending on the units employed, therefore $$d\frac{C}{2T\alpha^3} = \frac{B}{T\alpha}S\alpha\alpha',$$

or $$B = \tfrac{3}{2}\frac{C}{T\alpha^4},$$

and the resolved effect

$$= \frac{C\alpha\alpha_1}{2T\alpha_1}d\frac{S^2\alpha\alpha_1}{T\alpha^3} = C\alpha\alpha_1\frac{S\alpha\alpha_1}{T\alpha_1 T\alpha^5}(-\alpha^2 S\alpha'\alpha_1 + \tfrac{3}{2}S\alpha\alpha' S\alpha\alpha_1)$$

$$= C\alpha\alpha_1\frac{S\alpha\alpha_1}{T\alpha_1 T\alpha^5}(S.V\alpha\alpha' V\alpha\alpha_1 + \tfrac{1}{2}S\alpha\alpha' S\alpha\alpha_1).$$

The factor in brackets is evidently proportional in the ordinary notation to

$$\sin\theta\sin\theta'\cos\omega - \tfrac{1}{2}\cos\theta\cos\theta'$$

402. Thus the whole force is

$$\frac{C\alpha\alpha_1\alpha}{2S\alpha\alpha_1}d\frac{S^2\alpha\alpha_1}{T\alpha^3} = \frac{C\alpha\alpha_1\alpha}{2S\alpha\alpha'}d_1\frac{S^2\alpha\alpha'}{T\alpha^3},$$

as we should expect, $d_1\alpha$ being $= \alpha_1$. [This may easily be transformed into

$$-\frac{2C\alpha\alpha_1 U\alpha}{(T\alpha)^{\frac{1}{2}}}dd_1(T\alpha)^{\frac{1}{2}},$$

which is the quaternion expression for Ampère's well-known form.]

403. The whole effect on α_1 of the closed circuit, of which α' is an element, is therefore

$$\frac{C\alpha\alpha_1}{2}\int\frac{\alpha}{S\alpha\alpha_1}\,d\,\frac{(S\alpha\alpha_1)^2}{T\alpha^3},$$

$$=\frac{C\alpha\alpha_1}{2}\left\{\frac{\alpha S\alpha\alpha_1}{T\alpha^3}-V.\alpha_1\int\frac{V\alpha\alpha'}{T\alpha^3}\right\}$$

between proper limits. As the integrated part is the same at both limits, the effect is

$$-\frac{C\alpha\alpha_1}{2}V\alpha_1\beta,\text{ where }\beta=\int\frac{V\alpha\alpha'}{T\alpha^3}=\int\frac{dU\alpha}{\alpha},$$

and depends on the form of the closed circuit.

404. This vector β, which is of great importance in the whole theory of the effects of closed or indefinitely extended circuits, corresponds to the line which is called by Ampère "*directrice de l'action électrodynamique.*" It has a definite value at each point of space, independent of the existence of any other current.

Consider the circuit a polygon whose sides are indefinitely small; join its angular points with any assumed point, erect at the latter, perpendicular to the plane of each elementary triangle so formed, a vector whose length is $\frac{\omega}{r}$, where ω is the vertical angle of the triangle and r the length of one of the containing sides; the sum of such vectors is the "*directrice*" at the assumed point.

405. The mere *form* of the result of § 403 shows at once that *if the element* α_1 *be turned about its middle point, the direction of the resultant action is confined to the plane whose normal is* β.

Suppose that the element α_1 is forced to remain perpendicular to some given vector δ, we have

$$S\alpha_1\delta=0,$$

and the whole action in its plane of motion is proportional to $TV.\delta V\alpha_1\beta$.

But $$V.\delta V\alpha_1\beta = -\alpha_1 S\beta\delta.$$

Hence the action is evidently constant for all possible positions of α_1; or

The effect of any system of closed currents on an element of a conductor which is restricted to a given plane is (*in that plane*) *independent of the direction of the element.*

406. Let the closed current be *plane* and *very small.* Let ϵ (where $T\epsilon = 1$) be its normal, and let γ be the vector of any point within it (as the centre of inertia of its area); the middle point of α_1 being the origin of vectors.

Let $$\alpha = \gamma+\rho; \text{ therefore } \alpha' = \rho',$$

and $$\beta = \int \frac{V\alpha\alpha'}{T\alpha^3} = \int \frac{V(\gamma+\rho)\rho'}{T(\gamma+\rho)^3}$$

$$= \frac{1}{T\gamma^3} \int V(\gamma+\rho)\rho' \left\{1 + \frac{3\, S\gamma\rho}{T\gamma^2}\right\}$$

to a sufficient approximation.

Now (between limits)

$$\int V\rho\rho' = 2A\epsilon,$$

where A is the area of the closed circuit.

Also generally

$$\int V\gamma\rho' S\gamma\rho = \tfrac{1}{2}(S\gamma\rho\, V\gamma\rho + \gamma V.\gamma \int V\rho\rho')$$

$$= \text{(between limits) } A\gamma\, V\gamma\epsilon.$$

Hence for this case

$$\beta = \frac{A}{T\gamma^3}\left(2\epsilon + \frac{3\gamma\, V\gamma\epsilon}{T\gamma^2}\right)$$

$$= -\frac{A}{T\gamma^3}\left(\epsilon + \frac{3\gamma\, S\gamma\epsilon}{T\gamma^2}\right).$$

407. If, instead of one small plane closed current, there be a series of such, of equal area, disposed regularly in a tubular form, let x be the distance between two consecutive currents measured along the axis of the tube; then, putting $\gamma' = x\epsilon$, we have for the whole effect of such a set of currents on α_1

$$\frac{CA\alpha\alpha_1}{2x} V.\alpha_1 \int \left(\frac{\gamma'}{T\gamma^3} + \frac{3\gamma S\gamma\gamma'}{T\gamma^5} \right)$$

$$= \frac{CA\alpha\alpha_1}{2x} \frac{V\alpha_1\gamma}{T\gamma^3} \text{ (between proper limits).}$$

If the axis of the tubular arrangement be a closed curve this will evidently vanish. Hence *a closed solenoid exerts no influence on an element of a conductor. The same is evidently true if the solenoid be indefinite in both directions.*

If the axis extend to infinity in one direction, and γ_0 be the vector of the other extremity, the effect is

$$\frac{CA\alpha\alpha_1}{2x} \frac{V\alpha_1\gamma_0}{T\gamma_0^3},$$

and is therefore *perpendicular to the element and to the line joining it with the extremity of the solenoid. It is evidently inversely as* $T\gamma_0^2$ *and directly as the sine of the angle contained between the direction of the element and that of the line joining the latter with the extremity of the solenoid. It is also inversely as* x, *and therefore directly as the number of currents in a unit of the axis of the solenoid.*

408. To find the effect of the whole circuit whose element is α_1 on the extremity of the solenoid, we must change the sign of the above and put $\alpha_1 = \gamma_0'$; therefore the effect is

$$-\frac{CA\alpha\alpha_1}{2x} \int \frac{V\gamma_0'\gamma_0}{T\gamma_0^3},$$

an integral of the species considered in § 403 whose value is easily assigned in particular cases.

409. *Suppose the conductor to be straight, and indefinitely extended in both directions.*

Let $h\theta$ be the vector perpendicular to it from the extremity of the canal, and let the conductor be $\parallel \eta$, where $T\theta = T\eta = 1$.

Therefore $\gamma_0 = h\theta + y\eta$ (where y is a scalar),

$$V\gamma_0{}'\gamma_0 = hy' V\eta\theta,$$

and the integral in § 407 is

$$hV\eta\theta \int_{-\infty}^{+\infty} \frac{y'}{(h^2+y^2)^{\frac{3}{2}}} = \frac{2}{h} V\eta\theta.$$

The whole effect is therefore

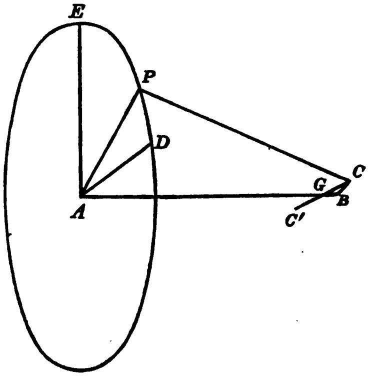
$$-\frac{CAaa_1}{xh} V\eta\theta,$$

and is thus *perpendicular to the plane passing through the conductor and the extremity of the canal, and varies inversely as the distance of the latter from the conductor.*

This is exactly the observed effect of an indefinite straight current on a magnetic pole, or particle of free magnetism.

410. *Suppose the conductor to be circular, and the pole nearly in its axis.*

Let EPD be the conductor, AB its axis, and C the pole; BC perpendicular to AB, and small in comparison with $AE = h$ the radius of the circle.



Let

AB be $a_1 i$, $\quad BC = bk$,

$AP = h(jx + ky)$

where $\left.\begin{matrix} x \\ y \end{matrix}\right\} = \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} \angle EAP = \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} \theta.$

Then $CP = \gamma = a_1 i + bk - h(jx + ky).$

And the effect on $C \propto \int \frac{V\gamma\gamma'}{T\gamma^3}$,

$$\propto h \int \frac{\theta' \{(h - by)i + a_1 xj + a_1 yk\}}{(a_1^2 + b^2 + h^2 - 2bhy)^{\frac{3}{2}}},$$

where the integral extends to the whole circuit.

411. Suppose in particular C to be one pole of a small magnet or solenoid CC' whose length is $2l$, and whose middle point is at G and distant a from the centre of the conductor. Let $\angle\, CGB = \Delta$. Then evidently

$$a_1 = a + l\cos\Delta,$$
$$b = l\sin\Delta.$$

Also the effect on C becomes, if $a_1^2 + b^2 + h^2 = A^2$,

$$\frac{h}{A^3}\int\theta'\{(h-by)i + a_1xj + a_1yk\}\{1 + \frac{3hby}{A^2} + \frac{15}{2}\frac{h^2b^2y^2}{A^4} + \dots\}$$

$$= \frac{\pi h^2}{A^3}\Big(2i - \frac{3b^2i}{A^2} + \frac{3a_1bk}{A^2} + \frac{15}{2}\frac{h^2b^2i}{A^4} + \dots\Big),$$

since for the whole circuit

$$\int\theta'y^{2n} = 2\pi\frac{\underline{|2n}}{2^{2n}(\underline{|n})^2},$$
$$\int\theta'y^{2n+1} = 0,$$
$$\int\theta'xy^m = 0.$$

If we suppose the centre of the magnet fixed, the vector axis of the couple produced by the action of the current on C is

$$lV.(i\cos\Delta + k\sin\Delta)\int\frac{V\gamma\gamma'}{T\gamma^3}$$

$$\propto \frac{\pi h^2 l\sin\Delta}{A^3}j\Big\{2 - \frac{3b^2}{A^2} + \frac{15}{2}\frac{h^2b^2}{A^4} - \frac{3a_1b\cos\Delta}{A^2\sin\Delta}\Big\}.$$

If A, &c. be now developed in powers of l, this at once becomes

$$\frac{\pi h^2 l\sin\Delta}{(a^2+h^2)^{\frac{3}{2}}}j\Big\{2 - \frac{6al\cos\Delta}{a^2+h^2} + \frac{15a^2l^2\cos^2\Delta}{(a^2+h^2)^2} - \frac{3l^2}{a^2+h^2}$$
$$- \frac{3l^2\sin^2\Delta}{a^2+h^2} + \frac{15}{2}\frac{h^2l^2\sin^2\Delta}{(a^2+h^2)^2} - 3\frac{(a+l\cos\Delta)l\cos\Delta}{a^2+h^2}\Big(1 - \frac{5al\cos\Delta}{a^2+h^2}\Big)\Big\}.$$

Putting $-l$ for l and changing the sign of the whole to get that for pole C', we have for the vector axis of the complete couple

$$\frac{4\pi h^2 l\sin\Delta}{(a^2+h^2)^{\frac{3}{2}}}j\Big\{1 + \tfrac{3}{4}\frac{l^2(4a^5 - h^2)(4 - 5\sin^2\Delta)}{(a^2+h^2)^2} + \text{\&c.}\Big\}$$

which is almost exactly proportional to $\sin\Delta$ if $2a = h$ and l be small.

On this depends a modification of the tangent galvanometer. (Bravais—*Ann. de Chimie,* xxxviii. 309.)

412. As before, the effect of an indefinite solenoid on α_1 is

$$\frac{CAaa_1}{2x}\,\frac{V\alpha_1\gamma}{T\gamma^3}.$$

Now suppose α_1 to be an element of a small plane circuit, δ the vector of the centre of inertia of its area, the pole of the solenoid being origin.

Let $\qquad \gamma = \delta+\rho,$ then $\alpha_1 = \rho'.$

The whole effect is therefore

$$-\frac{CAaa_1}{2x}\int\frac{V(\delta+\rho)\rho'}{T(\delta+\rho)^3}$$

$$=\frac{CAA_1aa_1}{2xT\delta^3}\left(\epsilon_1+\frac{3\delta S\delta\epsilon_1}{T\delta^2}\right),$$

where A_1 and ϵ_1 are, for the new circuit, what A and ϵ were for the former.

Let the new circuit also belong to an indefinite solenoid, and let δ_0 be the vector joining the poles of the two solenoids. Then the mutual effect is

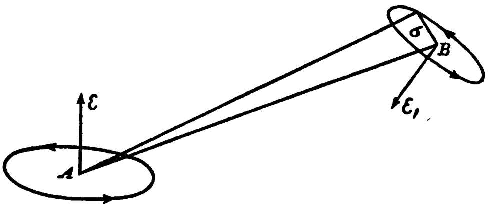
$$\frac{CAA_1aa_1}{2xx_1}\int\left(\frac{\delta'}{T\delta^3}+\frac{3\delta S\delta\delta'}{T\delta^5}\right)$$

$$=\frac{CAA_1aa_1}{2xx_1}\,\frac{\delta_0}{(T\delta_0)^3}\propto\frac{U\delta_0}{(T\delta_0)^2},$$

which is exactly *the mutual effect of two magnetic poles. Two finite solenoids, therefore, act on each other exactly as two magnets, and the pole of an indefinite solenoid acts as a particle of free magnetism.*

413. The mutual attraction of two indefinitely small plane closed circuits, whose normals are ϵ and ϵ_1, may evidently be deduced by twice differentiating the expression $\frac{U\delta}{T\delta^2}$ for the mutual action of the poles of two indefinite solenoids, making $d\delta$ in one differentiation $\parallel \epsilon$ and in the other $\parallel \epsilon_1$.

But it may also be calculated directly by a process which will give us in addition the couple impressed on one of the circuits by the other, supposing for simplicity the first to be *circular.*



Let A and B be the centres of inertia of the areas of A and B, ϵ and ϵ_1 vectors normal to their planes, σ any vector radius of B, $AB = \beta$.

Then whole effect on σ', by §§ 406, 403,

$$\propto \frac{A}{T(\beta+\sigma)^3} V\sigma' \left\{ \epsilon + \frac{3(\beta+\sigma)S(\beta+\sigma)\epsilon}{T(\beta+\sigma)^2} \right\},$$

$$\propto \frac{1}{T\beta^3} \left\{ V\sigma'\epsilon \left(1 + \frac{3S\beta\sigma}{T\beta^2}\right) + \frac{3V\sigma'\beta S\beta\epsilon}{T\beta^2}\left(1 + \frac{5S\beta\sigma}{T\beta^2}\right) \right.$$
$$\left. + \frac{3V\sigma'\beta S\sigma\epsilon}{T\beta^2} + 3\frac{V\sigma'\sigma S\beta\epsilon}{T\beta^2} \right\}.$$

But between proper limits,

$$\int V\sigma'\eta S\theta\sigma = -A_1 V.\eta V\theta\epsilon_1,$$

for generally $\int V\sigma'\eta S\theta\sigma = -\frac{1}{2}(V\eta\sigma S\theta\sigma + V.\eta V.\theta \int V\sigma\sigma')$.

Hence, after a reduction or two, we find that the whole force exerted by A on the centre of inertia of the area of B

$$\propto \frac{AA_1}{T\beta^5} \left\{ \beta \left(S\epsilon\epsilon_1 + \frac{5S\beta\epsilon S\beta\epsilon_1}{T\beta^2} \right) + \epsilon S\beta\epsilon_1 + \epsilon_1 S\beta\epsilon \right\}.$$

This, as already observed, may be at once found by twice differentiating $\frac{U\beta}{T\beta^2}$. In the same way the vector moment, due to A, about the centre of inertia of B,

$$\propto \frac{A}{T\beta^3}\int V.\sigma\left(V\sigma'\epsilon + \frac{3V\sigma'\beta S\beta\epsilon}{T\beta^2}\right),$$

$$\propto -\frac{AA_1}{T\beta^3}\left(V\epsilon\epsilon_1 + \frac{3V\beta\epsilon_1 S\beta\epsilon}{T\beta^2}\right).$$

These expressions for the whole force of one small magnet on the centre of inertia of another, and the couple about the latter, seem to be the simplest that can be given. It is easy to deduce from them the ordinary forms. For instance, the whole resultant couple on the second magnet

$$\propto \frac{T\left(V\epsilon\epsilon_1 + \frac{3V\beta\epsilon_1 S\beta\epsilon}{T\beta^2}\right)}{T\beta^3},$$

may easily be shown to coincide with that given by Ellis (*Camb. Math. Journal*, iv. 95), though it seems to lose in simplicity and capability of interpretation by such modifications.

414. The above formulae show that the whole force exerted by one small magnet M, on the centre of inertia of another m, consists of four terms which are, in order,

1st. *In the line joining the magnets, and proportional to the cosine of their mutual inclination.*

2nd. *In the same line, and proportional to five times the product of the cosines of their respective inclinations to this line.*

3rd. and 4th. *Parallel to* $\left\{\begin{matrix}\mathrm{m}\\ \mathrm{M}\end{matrix}\right\}$ *and proportional to the cosine of the inclination of* $\left\{\begin{matrix}\mathrm{M}\\ \mathrm{m}\end{matrix}\right\}$ *to the joining line.*

All these forces are, in addition, inversely as the fourth power of the distance between the magnets.

For the couples about the centre of inertia of *m* we have

1st. *A couple whose axis is perpendicular to each magnet, and which is as the sine of their mutual inclination.*

2nd. *A couple whose axis is perpendicular to* m *and to the line joining the magnets, and whose moment is as three times the product of the sine of the inclination of* m, *and the cosine of the inclination of* M, *to the joining line.*

In addition these couples vary inversely as the third power of the distance between the magnets.

[These results afford a good example of what has been called the *internal* nature of the methods of quaternions, reducing, as they do at once, the forces and couples to others independent of any lines of reference, other than those necessarily belonging to the system under consideration. To show their ready applicability, let us take a Theorem due to Gauss.]

415. *If two small magnets be at right angles to each other, the moment of rotation of the first is approximately twice as great when the axis of the second passes through the centre of the first, as when the axis of the first passes through the centre of the second.*

In the first case $\epsilon \parallel \beta \perp \epsilon_1$;

therefore moment $= \dfrac{C'}{T\beta^3} T(\epsilon\epsilon_1 - 3\epsilon\epsilon_1) = \dfrac{2C'}{T\beta^3} T\epsilon\epsilon_1$.

In the second $\epsilon_1 \parallel \beta \perp \epsilon$;

therefore moment $= \dfrac{C'}{T\beta^3} T\epsilon\epsilon_1$. Hence the theorem.

416. Again, we may easily reproduce the results of § 413, if for the two small circuits we suppose two small magnets perpendicular to their planes to be substituted. β is then the vector joining the middle points of these magnets, and by changing the tensors we may take 2ϵ and $2\epsilon_1$ as the vector lengths of the magnets.

Hence evidently the mutual effect

$$\propto \frac{U}{T^2}(\beta+\epsilon-\epsilon_1)-\frac{U}{T^2}(\beta-\epsilon-\epsilon_1)+\frac{U}{T^2}(\beta-\epsilon+\epsilon_1)-\frac{U}{T^2}(\beta+\epsilon+\epsilon_1),$$

which is easily reducible to

$$-\frac{12}{T\beta^5}\left\{\beta\left(S\epsilon\epsilon_1+\frac{5S\beta\epsilon S\beta\epsilon_1}{T\beta^2}\right)+\epsilon_1 S\beta\epsilon+\epsilon S\beta\epsilon_1\right\},$$

as before, if smaller terms be omitted.

If we operate with $V.\epsilon_1$ on the two first terms of the unreduced expression, and take the difference between this result and the same with the sign of ϵ_1 changed, we have the whole vector axis of the couple on the magnet $2\epsilon_1$, which is therefore, as before, seen to be proportional to

$$\frac{4}{T\beta^3}\left(V\epsilon_1\epsilon+\frac{3V\epsilon_1\beta S\beta\epsilon}{T\beta^2}\right).$$

417. We might apply the foregoing formulae with great ease to other cases treated by Ampère, De Montferrand, &c.—or to two finite circular conductors as in Weber's Dynamometer—but in general the only difficulty is in the integration, which even in some of the simplest cases involves elliptic functions, &c., &c.

418. Let $F(\gamma)$ be the potential of any system upon a unit particle at the extremity of γ.

$$F(\gamma)=C \quad\ldots\ldots\ldots\ldots \quad (1)$$

is the equation of a level surface.

Let the differential of (1) be

$$S\nu d\gamma=0, \quad\ldots\ldots\ldots\ldots \quad (2)$$

then ν is a vector normal to (1), and is therefore the *direction* of the force.

But, passing to a proximate level surface, we have

$$S\nu\delta\gamma=\delta C.$$

Make $\delta\gamma=x\nu$, then

$$-xT\nu^2=\delta C,$$

or

$$-T\nu=\frac{\delta C}{T\delta\gamma}.$$

Hence ν expresses the force in *magnitude* also. (§ 363.)

Now by § 406 we have for the vector force exerted by a small plane closed circuit on a particle of free magnetism the expression

$$-\frac{A}{T\gamma^3}\left(\epsilon+\frac{3\gamma S\gamma\epsilon}{T\gamma^2}\right),$$

omitting the factors depending on the strength of the current and the strength of magnetism of the particle.

Hence the potential, by (2) and (1),

$$\propto A\int\frac{1}{T\gamma^3}\left(S\epsilon d\gamma+\frac{3\,S\gamma d\gamma\, S\gamma\epsilon}{T\gamma^2}\right),$$

$$\propto\frac{AS\epsilon\gamma}{T\gamma^3},$$

$$\propto\frac{\text{area of circuit projected perpendicular to }\gamma}{T\gamma^2},$$

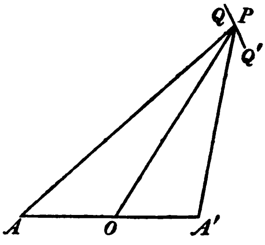
$\propto$ solid angle subtended by circuit.

The constant is omitted in the integration, as the potential must evidently vanish for infinite values of $T\gamma$.

By means of Ampère's idea of breaking up a finite circuit into an indefinite number of indefinitely small ones, it is evident that the above result may be at once extended to the case of such a finite closed circuit.

419. Quaternions give a simple method of deducing the well-known property of the *Magnetic Curves*.

Let A, A' be two equal magnetic poles, whose vector distance, 2α, is bisected in O, QQ' an indefinitely small magnet whose length is $2\rho'$, where $\rho = OP$. Then evidently, taking moments,



$$\frac{V(\rho+\alpha)\rho'}{T(\rho+\alpha)^3}=\pm\frac{V(\rho-\alpha)\rho'}{T(\rho-\alpha)^3}.$$

Operate by $S.V\alpha\rho$,

$$\frac{S\alpha\rho'(\rho+\alpha)^2 - S\alpha(\rho+\alpha)S\rho'(\rho+\alpha)}{T(\rho+\alpha)^3} = \pm \{\text{same with} -\alpha\},$$

or $$S.\alpha V\left(\frac{\rho'}{\rho+\alpha}\right)U(\rho+\alpha) = \pm \{\text{same with } -\alpha\},$$

i. e. $$S\alpha d\,U(\rho+\alpha) = \pm\, S\alpha d\,U(\rho-\alpha),$$

$$S\alpha\{U(\rho+\alpha) \mp U(\rho-\alpha)\} = \text{const.},$$

or $$\cos \angle OAP \pm \angle OA'P = \text{const.},$$

the property referred to.

420. If the vector of any point be denoted by

$$\rho = ix + jy + kz, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

there are many physically interesting and important transformations depending upon the effects of the quaternion operator

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

on various functions of ρ. When the function of ρ is a scalar, the effect of ∇ is to give the vector of most rapid increase. Its effect on a vector function is indicated briefly in § 364.

421. We commence with one or two simple examples, which are not only interesting, but very useful in transformations.

$$\nabla\rho = \left(i\frac{d}{dx} + \&c.\right)(ix + \&c.) = -3 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\nabla T\rho = \left(i\frac{d}{dx} + \&c.\right)(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{ix+jy+kz}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{\rho}{T\rho} = U\rho \ldots (4)$$

$$\nabla(T\rho)^n = n(T\rho)^{n-1}\nabla T\rho = n(T\rho)^{n-2}\rho; \quad\ldots\ldots\ldots\ldots \quad (5)$$

and, of course, $$\nabla\frac{1}{(T\rho)^n} = -\frac{n\rho}{(T\rho)^{n+2}}; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)^1$$

whence, $$\nabla\frac{1}{T\rho} = -\frac{\rho}{T\rho^3} = -\frac{U\rho}{T\rho^2}, \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

and, of course, $$\nabla^2 \frac{1}{T\rho} = -\nabla \frac{U\rho}{T\rho^2} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)^1$$

Also, $$\nabla\rho = -3 = T\rho\nabla U\rho + \nabla T\rho . U\rho = T\rho\nabla U\rho - 1,$$

$$\therefore \quad \nabla U\rho = -\frac{2}{T\rho}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

422. By the help of the above results, of which (6) is especially useful (though obvious on other grounds), and (4) and (7) very remarkable, we may easily find the effect of ∇ upon more complex functions.

Thus, $$\nabla S\alpha\rho = -\nabla(\alpha x + \&c.) = -\alpha, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\nabla V\alpha\rho = -\nabla V\rho\alpha = -\nabla(\rho\alpha - S\alpha\rho) = 3\alpha - \alpha = 2\alpha. \quad \ldots\ldots \quad (2)$$

Hence

$$\nabla \frac{V\alpha\rho}{T\rho^3} = \frac{2\alpha}{T\rho^3} - \frac{3\rho V\alpha\rho}{T\rho^5} = -\frac{2\alpha\rho^2 + 3\rho V\alpha\rho}{T\rho^5} = \frac{\alpha\rho^2 - 3\rho S\alpha\rho}{T\rho^5}. \quad (3)$$

Hence $$S.\delta\rho\nabla \frac{V\alpha\rho}{T\rho^3} = \frac{\rho^2 S\alpha\delta\rho - 3S\alpha\rho S\rho\delta\rho}{T\rho^5} = -\frac{S\alpha\delta\rho}{T\rho^3} - \frac{3S\alpha\rho S\rho\delta\rho}{T\rho^5}$$

$$= -\delta\frac{S\alpha\rho}{T\rho^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

This is a very useful transformation in various physical applications. By (6) it can be put in the sometimes more convenient form

$$S.\delta\rho\nabla\frac{V\alpha\rho}{T\rho^3} = \delta S.\alpha\nabla\frac{1}{T\rho}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

And it is worthy of remark that, as may easily be seen, $-S$ may be put for V in the left-hand member of the equation.

423. We have also

$$\nabla V.\beta\rho\gamma = \nabla\{\beta S\gamma\rho - \rho S\beta\gamma + \gamma S\beta\rho\} = -\gamma\beta + 3S\beta\gamma - \beta\gamma = S\beta\gamma. \quad (1)$$

Hence, if ϕ be any linear and vector function of the form

$$\phi\rho = \alpha + \Sigma V.\beta\rho\gamma + m\rho, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

i. e. a self-conjugate function with a constant vector added, then

$$\nabla\phi\rho = \Sigma S\beta\gamma - 3m = \text{scalar}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Hence, an integral of

$$\nabla\sigma = \text{scalar constant, is } \sigma = \phi\rho. \qquad \text{.............. (4)}$$

If the constant value of $\nabla\sigma$ contain a vector part, there will be terms of the form $V\epsilon\rho$ in the expression for σ, which will then express a distortion accompanied by rotation. (§ 366.)

Also, a solution of $\nabla q = a$ (where q and a are quaternions) is

$$q = S\zeta\rho + V\epsilon\rho + \phi\rho.$$

It may be remarked also, as of considerable importance in physical applications, that, by (1) and (2) of § 422,

$$\nabla(S + \tfrac{1}{2}V)a\rho = 0,$$

but we cannot here enter into details on this point.

424. It would be easy to give many more of these transformations, which really present no difficulty; but it is sufficient to show the ready applicability to physical questions of one or two of those already obtained; a property of great importance, as extensions of mathematical physics are far more valuable than mere analytical or geometrical theorems.

Thus, if σ be the vector-displacement of that point of a homogeneous elastic solid whose vector is ρ, we have, p being the consequent pressure produced,

$$\nabla p + \nabla^2\sigma = 0, \qquad \text{.......................... (1)}$$

whence $S\delta\rho\nabla^2\sigma = -S\delta\rho\nabla p = \delta p$, a complete differential. ... (2)

Also, generally, $p = kS\nabla\sigma,$

and if the solid be incompressible

$$S\nabla\sigma = 0. \qquad \text{............................ (3)}$$

Thomson has shown (*Camb. and Dub. Math. Journal*, ii. p. 62), that the forces produced by given distributions of matter, electricity, magnetism, or galvanic currents, can be represented at every point by displacements of such a solid producible by external forces. It may be useful to give his analysis, with some additions, in a quaternion form, to show the insight gained by the simplicity of the present method.

425. Thus, if $S\sigma\delta\rho = \delta\frac{1}{T\rho}$ we may write, each equal to

$$-S\delta\rho\nabla\frac{1}{T\rho}.$$

This gives $$\sigma = -\nabla\frac{1}{T\rho},$$

the vector-force exerted by one particle of matter or free electricity on another. This value of σ evidently satisfies (2) and (3).

Again, if

$$S.\delta\rho\nabla\sigma = \delta\frac{S\alpha\rho}{T\rho^3}, \text{ either is equal to}$$

$$-S.\delta\rho\nabla\frac{V\alpha\rho}{T\rho^3} \quad \text{by (4) of § 422.}$$

Here a particular case is

$$\sigma = -\frac{V\alpha\rho}{T\rho^3},$$

which is the vector-force exerted by an element α of a current upon a particle of magnetism at ρ. (§ 407.)

426. Also, by § 422 (3),

$$\nabla\frac{V\alpha\rho}{T\rho^3} = \frac{\alpha\rho^2 - 3\rho S\alpha\rho}{T\rho^5},$$

and we see by §§ 406, 407 that this is the vector-force exerted by a small plane current at the origin (its plane being perpendicular to α) upon a magnetic particle, or pole of a solenoid, at ρ. This expression, being a pure vector, denotes an elementary rotation caused by the distortion of the solid, and it is evident that the above value of σ satisfies the equations (2), (3), and the distortion is therefore producible by external forces. Thus the effect of an element of a current on a magnetic particle is expressed directly by the displacement, while that of a small closed current or magnet is represented by the vector-axis of the rotation caused by the displacement.

427. Again, let

$$S\delta\rho\nabla^2\sigma = \delta\frac{S\alpha\rho}{T\rho^3}.$$

It is evident that σ satisfies (2), and that the right-hand side of the above equation may be written

$$-S.\delta\rho\nabla\frac{V\alpha\rho}{T\rho^3}.$$

Hence a particular case is

$$\nabla\sigma = -\frac{V\alpha\rho}{T\rho},$$

and this satisfies (3) also.

Hence the corresponding displacement is producible by external forces, and $\nabla\sigma$ is the rotation axis of the element at ρ, and is seen as before to represent the vector-force exerted on a particle of magnetism at ρ by an element α of a current at the origin.

428. It is interesting to observe that a particular value of σ in this case is

$$\sigma = -\tfrac{1}{2}\nabla S\alpha U\rho - \frac{\alpha}{T\rho},$$

as may easily be proved by substitution.

Again, if $$S\delta\rho\sigma = -\delta\frac{S\alpha\rho}{T\rho^3},$$

we have evidently $$\sigma = \nabla\frac{S\alpha\rho}{T\rho^3}.$$

Now, as $\frac{S\alpha\rho}{T\rho^3}$ is the potential of a small magnet α, at the origin, on a particle of free magnetism at ρ, σ is the resultant magnetic force, and represents also a possible distortion of the elastic solid by external forces, since $\nabla\sigma = \nabla^2\sigma = 0$, and thus (2) and (3) are both satisfied.

MISCELLANEOUS EXAMPLES.

1. The expression

$$V\alpha\beta V\gamma\delta + V\alpha\gamma V\delta\beta + V\alpha\delta V\beta\gamma$$

denotes a vector. What vector?

2. If two surfaces intersect along a common line of curvature, they meet at a constant angle.

3. By the help of the quaternion formulae of rotation, translate into a new form the solution (given in § 234) of the problem of inscribing in a sphere a closed polygon the directions of whose sides are given.

4. Express, in terms of the masses, and geocentric vectors of the sun and moon, the sun's disturbing force on the moon, and expand it to terms of the second order; pointing out the magnitudes and directions of the separate components.

(Hamilton, *Lectures*, p. 615.)

5. If $q = r^{\frac{1}{2}}$, show that

$$\begin{aligned}
2dq &= 2dr^{\frac{1}{2}} = \tfrac{1}{2}(dr + Kq\,dr\,q^{-1})Sq^{-1} = \tfrac{1}{2}(dr + q^{-1}dr\,Kq)Sq^{-1} \\
&= (drq + Kq\,dr)q^{-1}(q + Kq)^{-1} = (drq + Kq\,dr)(r + Tr)^{-1} \\
&= \frac{dr + Uq^{-1}dr\,Uq^{-1}}{Tq(Uq + Uq^{-1})} = \frac{dr.Uq + Uq^{-1}dr}{q(Uq + Uq^{-1})} = \frac{q^{-1}(Uq\,dr + dr\,Uq^{-1})}{Uq + Uq^{-1}} \\
&= \frac{q^{-1}(q\,dr + Tr\,dr\,q^{-1})}{Tq(Uq + Uq^{-1})} = \frac{dr\,Uq + Uq^{-1}dr}{Tq(1 + Ur)} = \frac{dr\,Kq^{-1} + q^{-1}dr}{1 + Ur} \\
&= \left\{dr + V.Vdr\frac{V}{S}q\right\}q^{-1} = \left\{dr - V.Vdr\frac{V}{S}q^{-1}\right\}q^{-1} \\
&= \frac{dr}{q} + V.V\frac{dr}{q}\,\frac{V}{S}q = \frac{dr}{q} - V.V\frac{dr}{q}\,\frac{V}{S}q^{-1} \\
&= drq^{-1} + V.Vq^{-1}Vdr\left(1 + \frac{V}{S}q^{-1}\right):
\end{aligned}$$

and give geometrical interpretations of these varied expressions for the same quantity. (*Ibid.* p. 628.)

6. Derive (4) of § 92 directly from (3) of § 91.

7. Find the successive values of the continued fraction

$$u_x = \left(\frac{j}{i+}\right)^x 0,$$

where i and j have their quaternion significations, and x has the values 1, 2, 3, &c. (*Lectures*, p. 645.)

8. If we have

$$u_x = \left(\frac{j}{i+}\right)^x c,$$

where c is a given quaternion, find the successive values.

For what values of c does u become constant? (*Ibid.* p. 652.)

9. What vector is given, in terms of two known vectors, by the relation

$$\rho^{-1} = \tfrac{1}{2}(\alpha^{-1} + \beta^{-1})?$$

Show that the origin lies on the circle which passes through the extremities of these three vectors.

10. What problem has its conditions stated in the following six equations, from which ξ, η, ζ are to be determined as scalar functions of x, y, z, or of

$$\rho = ix + jy + kz?$$

$$\nabla^2\xi = 0, \qquad \nabla^2\eta = 0, \qquad \nabla^2\zeta = 0, \qquad S\nabla\xi\nabla\eta = 0,$$

$$S\nabla\eta\nabla\zeta = 0, \qquad S\nabla\zeta\nabla\xi = 0,$$

where $$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

Show that (with a change of origin) the general solution of these equations may be put in the form

$$S\rho(\phi + f)^{-1}\rho = 1,$$

where ϕ is a self-conjugate linear and vector function, and ξ, η, ζ

are to be found respectively from the three values of f at any point by relations similar to those in Ex. 24 to Chapter IX. (See Lamé, *Journal de Mathematiques*, 1843.)

11. Hamilton, *Bishop Law's Premium Examination*, 1862.

(*a.*) If OABP be four points of space, whereof the three first are given, and not collinear; if also OA $= \alpha$, OB $= \beta$, OP $= \rho$; and if, in the equation

$$F\frac{\rho}{\alpha} = F\frac{\beta}{\alpha},$$

the characteristic of operation F be replaced by S, the locus of P is a plane. What plane?

(*b.*) In the same general equation, if F be replaced by V, the locus is an indefinite right line. What line?

(*c.*) If F be changed to K, the locus of P is a point. What point?

(*d.*) If F be made $= U$, the locus is an indefinite half-line, or ray. What ray?

(*e.*) If F be replaced by T, the locus is a sphere. What sphere?

(*f.*) If F be changed to TV, the locus is a cylinder of revolution. What cylinder?

(*g.*) If F be made TVU, the locus is a cone of revolution. What cone?

(*h.*) If SU be substituted for F, the locus is one sheet of such a cone. Of what cone? and which sheet?

(*i.*) If F be changed to VU, the locus is a pair of rays. Which pair?

12. (*Ibid.* 1863.)

(*a.*) The equation $S\rho\rho' + a^2 = 0$

expresses that ρ and ρ' are the vectors of two points P and P′, which are conjugate with respect to the sphere

$$\rho^2 + a^2 = 0;$$

or of which one is on the polar plane of the other.

(*b.*) Prove by quaternions that if the right line PP′, connecting two such points, intersect the sphere, it is cut harmonically thereby.

(*c.*) If P′ be a given external point, the cone of tangents drawn from it is represented by the equation,

$$(V\rho\rho')^2 = a^2(\rho - \rho')^2;$$

and the orthogonal cone, concentric with the sphere, by

$$(S\rho\rho')^2 + a^2\rho^2 = 0.$$

(*d.*) Prove and interpret the equation,

$$T(n\rho - a) = T(\rho - na), \text{ if } T\rho = Ta.$$

(*e.*) Transform and interpret the equation of the ellipsoid,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2.$$

(*f.*) The equation

$$(\kappa^2 - \iota^2)^2 = (\iota^2 + \kappa^2)S\rho\rho' + 2S\iota\rho\kappa\rho'$$

expresses that ρ and ρ' are values of conjugate points, with respect to the same ellipsoid.

(*g.*) The equation of the ellipsoid may also be thus written,

$$S\nu\rho = 1, \text{ if } (\kappa^2 - \iota^2)^2\nu = (\iota - \kappa)^2\rho + 2\iota S\kappa\rho + 2\kappa S\iota\rho.$$

(*h.*) The last equation gives also,

$$(\kappa^2 - \iota^2)^2\nu = (\iota^2 + \kappa^2)\rho^2 + 2V\iota\rho\kappa.$$

(*i.*) With the same signification of ν, the differential equations of the ellipsoid and its reciprocal become

$$S\nu d\rho = 0, \qquad S\rho d\nu = 0.$$

(*j.*) Eliminate ρ between the four scalar equations,

$$S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c, \quad S\epsilon\rho = e.$$

13. (*Ibid.* 1864.)

(*a.*) Let A_1B_1, A_2B_2, ... A_nB_n be any given system of posited right lines, the $2n$ points being all given; and let their vector sum,

$$AB = A_1B_1 + A_2B_2 + \ldots + A_nB_n,$$

be a line which does not vanish. Then a point H, and a scalar h, can be determined, which shall satisfy the quaternion equation,

$$HA_1 . A_1B_1 + \ldots + HA_n . A_nB_n = h . AB;$$

namely by assuming any origin O, and writing,

$$OH = V\frac{OA_1 . A_1B_1 + \ldots + OA_n . A_nB_n}{A_1B_1 + \ldots + A_nB_n},$$

$$h = S\frac{OA_1 . A_1B_1 + \ldots}{A_1B_1 + \ldots}.$$

(*b.*) For any assumed point C, let

$$Q_C = CA_1 . A_1B_1 + \ldots + CA_n . A_nB_n;$$

then this quaternion sum may be transformed as follows,

$$Q_C = Q_H + CH . AB = (h + CH) . AB;$$

and therefore its tensor is,

$$TQ_C = (h^2 + \overline{CH}^2)^{\frac{1}{2}} . \overline{AB},$$

in which $\overline{AB}$ and $\overline{CH}$ denote lengths.

(*c.*) The least value of this tensor TQ_C is obtained by placing the point C at H; if then a quaternion be said to be a minimum when its tensor is such, we may write

$$\text{min.}\, Q_C = Q_H = h . AB;$$

so that this minimum of Q_C is a vector.

(*d.*) The equation

$$TQ_C = c = \text{any scalar constant} > TQ_H$$

expresses that the locus of the variable point C is a spheric surface, with its centre at the fixed point H, and with a radius r, or $\overline{CH}$, such that

$$r.\overline{AB} = (TQ_C{}^2 - TQ_H{}^2)^{\frac{1}{2}} = (c^2 - h^2.\overline{AB}^2)^{\frac{1}{2}};$$

so that H, as being thus the common centre of a series of concentric spheres, determined by the given system of right lines, may be said to be the *Central Point*, or simply the *Centre*, of that system.

(*e.*) The equation

$$TVQ_C = c_1 = \text{any scalar constant} > TQ_H$$

represents a right cylinder, of which the radius $= (c_1^2 - h^2.\overline{AB}^2)^{\frac{1}{2}}$ divided by $\overline{AB}$, and of which the axis of revolution is the line,

$$VQ_C = Q_H = h.AB;$$

wherefore this last right line, as being the common axis of a series of such right cylinders, may be called the *Central Axis* of the system.

(*f.*) The equation

$$SQ_C = c_2 = \text{any scalar constant}$$

represents a plane; and all such planes are parallel to the *Central Plane*, of which the equation is

$$SQ_C = 0.$$

(*g.*) Prove that the central axis intersects the central plane perpendicularly, in the central point of the system.

(*h.*) When the n given vectors $A_1B_1, \dots A_nB_n$ are parallel, and are therefore proportional to n scalars, $b_1, \dots b_n$, the scalar h and the vector Q_H vanish; and the centre H is then determined by the equation,

$$b_1.HA_1 + b_2.HA_2 + \dots + b_n.HA_n = 0,$$

or by the expression,

$$OH = \frac{b_1.OA_1 + \dots + b_n.OA_n}{b_1 + \dots + b_n},$$

where O is again an arbitrary origin.

14. (*Ibid.* 1860.)

(*a.*) The normal at the end of the variable vector ρ, to the surface of revolution of the sixth dimension, which is represented by the equation

$$(\rho^2-\alpha^2)^3 = 27\alpha^2(\rho-\alpha)^4, \quad \dots\dots\dots\dots\dots\dots \quad \text{(a)}$$

or by the system of the two equations,

$$\rho^2-\alpha^2 = 3t^2\alpha^2, \quad (\rho-\alpha)^2 = t^3\alpha^2, \dots\dots\dots\dots\dots \text{(a}'\text{)}$$

and the tangent to the meridian at that point, are respectively parallel to the two vectors,

$$\nu = 2(\rho-\alpha)-t\rho,$$

and $$\tau = 2(1-2t)(\rho-\alpha)+t^2\rho;$$

so that they intersect the axis α, in points of which the vectors are, respectively,

$$\frac{2\alpha}{2-t}, \quad \text{and} \quad \frac{2(1-2t)\alpha}{(2-t)^2-2}.$$

(*b.*) If $d\rho$ be in the same meridian plane as ρ, then

$$t(1-t)(4-t)\,d\rho = 3\tau dt, \quad \text{and} \quad S\frac{\rho\,dt}{d\rho} = \frac{4-t}{3}.$$

(*c.*) Under the same condition,

$$S\frac{d\nu}{d\rho} = \frac{2}{3}(1-t).$$

(*d.*) The vector of the centre of curvature of the meridian, at the end of the vector ρ, is, therefore,

$$\sigma = \rho-\nu\left(S\frac{d\nu}{d\rho}\right)^{-1} = \rho-\frac{3}{2}\frac{\nu}{1-t} = \frac{6\alpha-(4-t)\rho}{2(1-t)}.$$

(*e.*) The expressions in Example 38 give

$$\nu^2 = \alpha^2t^2(1-t)^2, \quad \tau^2 = \alpha^2t^3(1-t)^2(4-t);$$

hence

$$(\sigma-\rho)^2 = \frac{9}{4}\alpha^2t^2, \quad \text{and} \quad d\rho^2 = \frac{9\alpha^2t}{4-t}dt^2;$$

the radius of curvature of the meridian is, therefore,

$$R = T(\sigma - \rho) = \frac{3}{2} t Ta;$$

and the length of an element of arc of that curve is

$$ds = Td\rho = 3\,Ta\left(\frac{t}{4-t}\right)^{\frac{1}{2}} dt.$$

(*f.*) The same expressions give

$$4(Va\rho)^2 = -a^4 t^3 (1-t)^2 (4-t);$$

thus the auxiliary scalar t is confined between the limits 0 and 4, and we may write $t = 2 \operatorname{vers} \theta$, where θ is a real angle, which varies continuously from 0 to 2π; the recent expression for the element of arc becomes, therefore,

$$ds = 3\,Ta.td\theta,$$

and gives by integration

$$s = 6\,Ta\,(\theta - \sin\theta),$$

if the arc s be measured from the point, say F, for which $\rho = a$, and which is common to all the meridians; and the total periphery of any one such curve is $= 12\pi Ta$.

(*g.*) The value of σ gives

$$4(\sigma^2 - a^2) = 3a^2 t (4-t), \quad 16(Va\sigma)^2 = -a^4 t^3 (4-t)^3;$$

if, then, we set aside the axis of revolution a, which is *crossed* by all the normals to the surface (a), the surface of centres of curvature which is *touched* by all those normals is represented by the equation,

$$4(\sigma^2 - a^2)^3 + 27a^2(Va\sigma)^2 = 0. \quad \ldots\ldots\ldots\ldots\ldots \text{(b)}$$

(*h.*) The point F is common to the two surfaces (a) and (b), and is a singular point on each of them, being a triple point on (a), and a double point on (b); there is also at it an infinitely sharp cusp on (b), which tends to coincide with the axis a, but a determined tangent plane to (a), which is perpendicular to that axis, and to that cusp; and the point, say F', of which the vector $= -a$, is another and an exactly similar cusp on (b), but does not belong to (a).

(*i.*) Besides the *three* universally *coincident* intersections of the surface (a), with *any* transversal, drawn through its triple point F, in *any* given direction β, there are always *three other real intersections*, of which indeed one coincides with F if the transversal be perpendicular to the axis, and for which the following is a general formula:

$$\rho = Ta.[Ua + \{2SU(a\beta)^{\frac{1}{3}}\}^3 U\beta].$$

(*j.*) The point, say V, of which the vector is $\rho = 2a$, is a double point of (a), near which that surface has a cusp, which coincides nearly with its tangent cone at that point; and the semi-angle of this cone is $= \frac{\pi}{6}$.

AUXILIARY EQUATIONS.

$$\begin{cases} 2S\rho(\rho-a) = a^2t^2(3+t), \\ 2Sa(\rho-a) = a^2t^2(3-t). \end{cases}$$

$$\begin{cases} S\nu\rho = -a^2t(1-t)(1-2t), \\ 2S\nu(\rho-a) = a^2t^3(1-t). \end{cases}$$

$$\begin{cases} S\rho\tau = a^2t^2(1-t)(4-t), \\ 2S(\rho-a)\tau = a^2t^3(1-t)(4-t). \end{cases}$$

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With one exception all writers before the Reformation are excluded from the Series. However great may be the value of literature before that epoch, it is not completely national. For it had no common organ of language; it addressed itself to special classes; it dealt mainly with special subjects. Again; of writers who flourished after the Reformation, who were popular in their day, and reflected the manners and sentiments of their age, the larger part by far must be excluded from our list. Common sense tells us that if young persons, who have but a limited time at their disposal, read Marlowe or Greene, Burton, Hakewill or Du Bartas, Shakspeare, Bacon, and Milton will be comparatively neglected.

Keeping, then, to the best authors in each epoch—and here popular estimation is a safe guide—the student will find the following list of writers amply sufficient for his purpose: Chaucer, Spenser, Hooker, Shakspeare, Bacon, Milton, Dryden, Bunyan, Pope, Johnson, Burke, and Cowper. In other words, Chaucer is the exponent of the Middle Ages in England; Spenser of the Reformation and the Tudors; Hooker of the latter years of Elizabeth; Shakspeare and Bacon of the transition from Tudor to Stuart; Milton of Charles I and the Commonwealth; Dryden and Bunyan of the Restoration; Pope of Anne and the House of Hanover; Johnson, Burke, and Cowper of the reign of George III to the close of the last century.

The list could be easily enlarged; the names of Jeremy Taylor, Clarendon, Hobbes, Locke, Swift, Addison, Goldsmith, and others are omitted. But in so wide a field, the difficulty is to keep the series from becoming unwieldly, without diminishing its comprehensiveness. Hereafter, should the plan prove to be useful, some of the masterpieces of the authors just mentioned may be added to the list.

The task of selection is not yet finished. For purposes of education, it would neither be possible, nor, if possible, desirable, to place in the hands of students the whole of the works of the authors we have chosen. We must set before them only the masterpieces of literature, and their studies must be directed, not only to the greatest minds, but to their choicest productions. These are to be read again and again, separately and in combination. Their purport, form, language, bearing on the times, must be minutely studied, till the student begins to recognise the full value of each work both in itself and in its relations to those that go before and those that follow it.

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